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Unpacking pre-service teachers' conceptualization of logarithmic differentiation through the APOS theory

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Abstract

Logarithmic differentiation is an effective method that aids the process of finding the derivatives of complex exponential functions. However, there has been a scarcity of studies, particularly in the South African context, that have provided evidence on pre-service mathematics teachers' understanding of the concept of logarithmic differentiation. This study explored pre-service teachers' conceptualization of logarithmic differentiation through action-process-object-schema (APOS) theory. We employed a qualitative case study design involving 90 first-year pre-service teachers enrolled in a mathematics teacher education program at a university in South Africa's Eastern Cape Province. Overall, the analysis showed that 63.9% of the participants demonstrated a substantial understanding of logarithmic differentiation processes, including 46.1% who had reached the schema stage. Nonetheless, common misconceptions and errors persisted, particularly among those who operated at action and beginning process stages. Errors and misconceptions such as the misapplication of differentiation rules, calculation errors in combining derivatives, and conceptual misunderstanding were evident. These findings highlight the need for mathematics teacher preparation programs to emphasize both conceptual and procedural understanding of differentiation. Achieving this goal may involve targeted instruction on relevant foundational concepts, continuous professional development, and integration of active learning strategies, such as the activities, classroom discussions, and exercises (ACE) teaching cycles to address common misconceptions.

Keywords: ACE teaching cycle, APOS theory, first-year students, logarithmic differentiation, pre-service teachers

INTRODUCTION

In the dynamic landscape of mathematics education, differential calculus stands as a pivotal construct in studying different forms of functions, rates of change, and dynamic processes. Its significance ripples across diverse fields, from physics and engineering to economics and computer science, among others (Feudel & Biehler, 2022; Jones, 2017; Mkhatshwa, 2018). As such, prospective secondary school mathematics teachers must not merely grasp derivatives but also recognize their essential role in shaping learners' mathematical understanding. Research shows that derivatives serve diverse purposes, some of which include modeling real-world phenomena as well as optimization and problem solving (Jones, 2017; Thompson & Harel, 2021). Derivatives and integrals are inseparable companions in the sense that the study of certain techniques of integration involves the direct application of derivatives (Balachandran, 2023).

Despite their fundamental importance, students often grapple with differential calculus. Recent research highlights common challenges faced by learners at both high school and college levels (Alam, 2020; Listiawati et al., 2023; Mkhatshwa, 2024). A study by Mkhatshwa (2020) reveals that students harbor misconceptions about derivatives, from confusing the chain rule with the product rule to misinterpreting the concept of limits. Another study by Siyepu (2013) from a South African context found that students commonly made errors in derivatives due to over-generalization of mathematical rules, such as the power rule and distributive property.

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Contribution to the literature

- This study suggests that the activities, classroom discussions, and exercises (ACE) teaching cycles positively influenced pre-service teachers' understanding of logarithmic differentiation, which aligns with other existing studies that reported significant improvements in cognitive engagement when using this instructional approach.
- The study identifies common misconceptions and errors among pre-service teachers, particularly the misapplication of differentiation rules and calculation mistakes. This aligns with prior research and emphasizes the need for addressing these issues in mathematics teacher education.
- Based on the findings, this study recommends incorporating targeted courses or additional professional development training sessions in teacher education curricula. These efforts can address misconceptions and enhance instructional practices.

Understanding these misconceptions and errors is crucial for effective teaching. Sánchez-Matamoros et al. (2015) suggest that pre-service teachers, who will soon teach differential calculus to high school students, need a robust understanding of this subject area. However, gaps in pre-service teachers' content knowledge and pedagogical strategies can hinder their ability to convey these concepts effectively. Moreover, there is a scarcity of studies, particularly in the South African context, and sub-Saharan Africa, in general, that have provided evidence on pre-service mathematics teachers' understanding of the concept of derivatives using natural logarithms.

The present study focuses on pre-service teachers' conceptualization of logarithmic differentiation using the action-process-object-schema (APOS) theory. By assessing pre-service teachers' grasp of this essential mathematical concept, this study provides insights into pre-service teachers' ability to use appropriate methods in solving differentiation problems requiring the application of the logarithmic technique. As this study focuses on the concept of "logarithmic differentiation," we recognize its place within the broader landscape of differential calculus as a terrain that encompasses power rules, chain rules, product rules, and implicit differentiation among other techniques.

LITERATURE REVIEW

Students' Conceptualization of Logarithmic Differentiation

Logarithmic differentiation is an effective method that aids the process of finding the derivatives of complex functions by applying the natural logarithm as well as leveraging logarithmic properties. It is particularly useful for functions involving products, quotients, or exponents (Mullan et al., 2015; Strang & Herman, 2016). Nonetheless, logarithmic differentiation depends on exponential and logarithmic rules, which must be fully understood (Díaz-Berrios & Martínez-Planell, 2022). Research indicates that students frequently make errors when dealing with exponents and logarithms due to underdeveloped conceptions of specific rules and properties (Cangelosi et al., 2013; Okoye-Ogbalu & Nnadozie, 2024). This issue is exacerbated by the fact that calculus often requires the use of logarithmic functions in its computations.

Although numerous studies have explored students' conceptualization of derivatives, few, if any, have specifically addressed pre-service teachers' understanding of logarithmic differentiation. Studies within South Africa (e.g., Naidoo & Naidoo, 2007; Siyepu, 2013, 2015) and outside South Africa (e.g., Jones, 2017; Listiawati et al., 2023; Mkhatshwa, 2023; Sánchez-Matamoros et al., 2015; Törner et al., 2014) have consistently documented students' difficulties in understanding derivatives, but none of these has specifically focused on logarithmic differentiation. Common errors and misconceptions include the misapplication of differentiation rules such as the power rule, chain rule, quotient rule, and product rule (Mkhatshwa, 2020). Students also tend to confuse different notations for derivatives, leading to their failure to recognize that notations like $f'(x), y'$ or $\frac{dy}{dx}$ represent the same concept.

Research also shows that errors in algebraic manipulations can hinder students' success in differential calculus (Orton, 1983). Luneta and Makonye (2010) observed that students' overreliance on procedural knowledge without a conceptual basis prevents them from generating correct solutions. This lack of conceptual understanding often leads students to mimic examples, which can result in incorrect answers, especially when the task differs from the worked example. Another reason for students' difficulties with derivatives is the prerequisite understanding of other concepts (Wille, 2017, 2020). Misconceptions about derivatives of trigonometric functions also impede students' success in differential calculus (Siyepu, 2015).

In the specific context of logarithmic differentiation, students must understand both logarithmic and exponential rules, and the appropriate methods for differentiating them (Mullan et al., 2015; Strang & Herman, 2016). This complexity can lead to difficulties, particularly with functions like $y = x^x$, where students might confuse the differentiation process with the common power rule, failing to recognize the need for logarithmic differentiation (Brannen & Ford, 2004). Additionally, students' inability to apply implicit differentiation rules to functions of the form $y = [f(x)]^{g(x)}$ can result in incorrect solutions. This is why it is essential to develop students' ability to make relevant connections between the mathematical concept of the derivative and its interpretation in various contexts (Feudel & Biehler, 2022; Thompson & Harel, 2021).

This scenario affirms the necessity of unpacking pre-service teachers' understanding of logarithmic differentiation to provide a foundation for targeted instruction and practice. Doing so will help mathematics teacher educators and training institutions offer pre-service teachers the opportunities to develop a deeper understanding of differential calculus concepts, particularly logarithmic differentiation, thereby enhancing their mathematical knowledge for teaching.

APOS Theory and Logarithmic Differentiation

APOS theory, rooted in Piaget's (1971) reflective abstraction, posits that mathematical understanding develops through reflecting on problems and constructing mental structures within schemas. Introduced by Dubinsky (1984) and abbreviated by Cottrill et al. (1996), APOS theory emphasizes a constructivist approach to learning in a social context. Although this theory was particularly introduced for understanding undergraduate students' conceptualization of mathematical concepts, it has also found its relevance in elementary and high school mathematics education research (Arnon et al., 2014; Dubinsky, 2014). It consists of four interconnected components namely action, process, object, and schema.

The "action" stage involves explicit steps taken to manipulate mathematical objects. Students perform these actions either physically or mentally to transform the given expressions. For example, when dealing with the derivative of a function such as $y = (4x - e^{3x})^{\tan x}$, an action might involve recognizing the need to introduce the natural logarithm to both sides of the equation. This recognition and physically manipulating the expression sets the groundwork for further differentiation. It is also worth noting that even if a student performs a wrong calculation or applies incorrect methods, it can still be classified as the "action" stage within the APOS theory.

In the "process" stage, learners perform the same operations as in the "action" stage, but entirely within their minds. These mental simulations begin to deepen conceptual understanding. For instance, after introducing the common logarithm to both sides (as outlined in the "action" stage), the "process" stage involves mentally applying logarithmic differentiation steps. Students imagine the application of logarithmic properties and differentiate without physically writing out each step. This internal mental process enhances their grasp of the concept.

The "object" stage incorporates processes, allowing students to manipulate mathematical entities. It requires an operational understanding of algebraic manipulations, such as using the power, sum, chain, product, and quotient rules appropriately. Having transformed the given function using logarithms, students may now treat the resulting expressions as objects. They manipulate these expressions using established differentiation rules. For instance, they might differentiate the logarithmic expression $\ln(4x - e^{3x})^{\tan x}$ using a combination of rules by first dropping the power to get $\tan x \ln(4x - e^{3x})$, and then perform the product rule, which in the process incorporates other rules such as differentiation of trigonometric and logarithmic functions, power rule, and chain rule.

Finally, schemas are coherent frameworks that organize and link actions, processes, and objects. A well-constructed schema enables strategic application of differentiation in various contexts. For example, students who have developed a robust schema for logarithmic differentiation can effortlessly apply it to diverse functions like $y = (4x - e^{3x})^{\tan x}$, recognizing when and how to use logarithmic properties and differentiation rules efficiently. Students are also able to adapt their approach based on the specific problem or context.

The relevance of APOS theory to this study lies in its ability to unpack pre-service teachers' conceptualization of logarithmic differentiation. By analyzing their understanding through the APOS framework, this study sought to identify at which level (action, process, object, or schema) pre-service teachers encounter difficulties and suggest instruction to address the identified gaps. This approach helps in developing a deeper understanding of differential calculus concepts, particularly logarithmic differentiation, thereby enhancing pre-service teachers' content knowledge.

APOS theory has been applied in various scenarios in mathematics education research, providing insights into students' understanding and misconceptions across different mathematical domains. For example, it has been used to study the learning of binomial expansions (Tatira, 2021), exponential and logarithmic functions (Díaz-Berrios & Martínez-Planell, 2022; Okoye-Ogbalu & Nnadozie, 2024), calculus (Borji et al., 2018; Moru, 2020; Siyepu, 2013, 2015), trigonometry (Nabie et al., 2018; Ngcobo et al., 2019; Walsh et al., 2017) and linear algebra (Mutambara & Bansilal, 2019; Tatira, 2023) among other topics. These studies have revealed that APOS theory can help teachers to design instructional strategies that foster deeper comprehension by moving students from actions to schemas, thereby ensuring a more holistic and interconnected understanding of mathematical concepts.

Scope of the Current Study

In the South African school curriculum, grade 12 learners are introduced to the basics of differentiation. According to the curriculum and assessment policy statement by the Department of Basic Education (DBE, 2011), grade 12 learners are expected to develop an intuitive understanding of limit concepts, differentiation from first principles, and the power rule. At this level, learners also encounter the second derivative, $\frac{d^2y}{dx^2}$, and its applications in determining concavity, graph sketching, and solving practical problems like optimization, rates of change, and calculus of motion.

At the university where this study was conducted, first-year education students training to become mathematics teachers learn additional aspects of differentiation, for example, the chain rule, quotient rule, implicit differentiation and inverse trigonometric functions. One topic introduced at the first-year level is logarithmic differentiation, which serves as the focus of this study. Beyond preparing these student teachers to teach calculus to secondary school learners upon qualification, they are also provided with a solid foundation for learning further integration techniques. This is important because differentiation rules often apply when dealing with integration (Strang & Herman, 2016). Additionally, it equips them with the necessary background to explore other areas of calculus, including differential equations, partial derivatives, and proofs in real analysis, among others.

As indicated earlier, this study employed APOS theory as an analytical framework for pre-service teachers' responses to a task on logarithmic differentiation. This topic was taught using the ACE teaching cycles. The ACE teaching cycle is an instructional approach well-aligned with APOS theory (Arnon et al., 2014). It consists of three components: ACE completed outside the class. These components are repeated cyclically, guiding students through the APOS stages. Activities engage students in mathematical actions (e.g., problem-solving, and exploration), while classroom discussions promote collaborative sense-making and deeper understanding. On the other hand, exercises reinforce learning and help build schemas by applying concepts.

In this study, ACE teaching cycles were used to help students conceptualize logarithmic differentiation, which was the last technique of differentiation to be taught in the first-year pre-calculus course. As in existing research (Abakah & Brijlall, 2024; Borji et al., 2018; Koyunkaya & Boz-Yaman, 2023), students were given numerous activities before and after classes. Classroom discussions refined and developed relevant mental structures according to APOS. These activities, discussions, and exercises were repeated at each stage until undergraduate students (pre-service teachers) displayed a certain level of understanding. Due to the scarcity of studies, particularly in South Africa, that have documented pre-service teachers' content knowledge of logarithmic differentiation, this study aimed to explore their conceptualization of this topic through APOS theory. Specifically, the following research questions were explored:

1. How do pre-service teachers' understanding of the differentiation of exponential functions manifest through the stages of APOS theory?
2. What common misconceptions and errors do pre-service teachers exhibit in the differentiation of complex exponential functions?

METHODOLOGY

This study was situated within the constructivist paradigm, which aligns with APOS theory, emphasizing learners' active engagement in constructing mathematical knowledge (Arnon et al., 2014; Dubinsky, 2014). This was particularly relevant in the sense that constructivism acknowledges the influence of context and individual differences on learning processes.

We employed a qualitative case study research design (Creswell, 2014). While we incorporated some elements of quantitative analysis to reveal patterns and the prevalence of errors, the study was predominantly qualitative. We aimed to provide an in-depth exploration of pre-service teachers' understanding of logarithmic differentiation. The adoption of a qualitative case study design is further justified by the fact that it enabled us to offer rich, context-specific insights by examining individual cases (Creswell & Creswell, 2018). Additionally, the APOS theory's emphasis on learners' active engagement aligns well with this design, which allowed us to explore mental processes, context, and individual differences

Our sample consisted of 90 first-year pre-service teachers enrolled in a mathematics teacher education program at a university in the Eastern Cape Province of South Africa. Participants were selected based on their enrollment in the specific mathematics teacher education program. We administered a test on logarithmic differentiation to all participants and analyzed their responses to the following test questions (test items):

1. Find the derivative of $f(x) = (2x - e^{8x})^{\sin 2x}$
2. Suppose $y = x^{\ln x}$, find y'.

While both items involve logarithmic differentiation, the first one is more complex due to the composite nature of the function and the trigonometric power. The second item is less complex but still requires thoughtful application of differentiation rules. Evaluating responses to these items allowed us to assess pre-service teachers' ability to handle non-routine scenarios related to logarithmic differentiation. The varying complexity of the two items also led to both moderate and high cognitive demands in terms of participants' mental constructions and reasoning.

Preliminary analysis of participants' solutions (responses) to the two items was done. At this stage, we categorized the responses into four groups, namely un-attempted, incorrect, partially correct, and correct responses[1]. While this study was predominantly qualitative, this categorization allowed us to summarize pre-service teachers' response patterns quantitatively using descriptive statistics such as frequencies and percentages.

Subsequently, we conducted a detailed content analysis of the participants' test scripts to unpack pre-service teachers' conceptualization of logarithmic differentiation through the APOS theory. Following the procedure outlined by (Braun & Clarke, 2006), we conducted thematic analysis to identify common errors and misconceptions related to logarithmic differentiation.

To gain deeper insights into pre-service teachers' mental constructions and reasoning, as well as their perceived difficulties with the concept of logarithmic differentiation, we also conducted interviews with 10 of the 90 participants. Among those selected for interviews, we ensured representation from each response category (unattempted, incorrect, partially correct, and correct). By integrating interview responses with the quantitative summaries and participants' written solutions to the two test items, we gained insights into pre-service teachers' conceptualization of logarithmic differentiation. To ensure anonymity, we assigned pseudonyms ranging from P001 to P090 to identify research participants during data analysis and reporting.

FINDINGS

Response Categories

In the analysis of test scripts from research participants, four distinct response categories emerged. First, the "unattempted" category refers to those who left a particular question blank or merely copied the question without providing a solution. Second, the "incorrect" category encompasses participants who attempted a question but failed to use appropriate methods or strategies to arrive at the correct answer. Third, the "partially correct" category applies to those who answered certain portions correctly but did not achieve a fully correct solution. Finally, the "correct" category indicates participants who answered a particular question correctly.

Based on the results presented in **Table 1**, most students attempted both items, with only 7.8% of responses unattempted for each item. This indicates that most students felt they had at least some understanding or were willing to try solving the problems. The highest proportion of responses falls in the "correct" category for both items, with 44.4% for item 1 and 47.8% for item 2. This suggests that a significant portion of the students have a good understanding of logarithmic differentiation.

Table 1. Response categories for both items

Response category	Item 1		Item 2	
	Count (n)	Percent	Count (n)	Percent
Unattempted	7	7.8	7	7.8
Incorrect	25	27.8	26	28.9
Partially correct	18	20.0	14	15.6
Correct	40	44.4	43	47.8
Total	90	100	90	100

For item 1, results show that 27.8% of the responses were incorrect, and 20.0% were partially correct. For item 2, it has been observed that 28.9% were incorrect, and 15.6% were partially correct. The similarity in these percentages suggests a consistent level of misunderstanding or partial understanding across the two items.

Results displayed in **Table 1** further illustrates that the percentage of correct responses is slightly higher for item 2 (47.8% compared to 44.4% for item 1), and the percentage of partially correct responses is slightly lower for item 2 (15.6% compared to 20.0% for item 1). This might suggest that item 2 was slightly easier for the students or that they were more familiar with the concepts tested by item 2 compared to item 1.

APOS Levels Attained and Identified Errors or Misconceptions

The findings presented in **Table 1** reveal different levels of pre-service teachers' understanding of logarithmic differentiation.

Pre-action stage

Firstly, for the 7.8% of responses that were unattempted, students might lack the fundamental understanding or confidence to begin the problem, placing them at the pre-action stage. This indicates that they have not yet internalized the initial steps or actions necessary to start the logarithmic differentiation process. This became apparent during interviews with some pre-service teachers who exhibited a high level of difficulty with logarithmic differentiation. The following conversation between the researcher and one of the participants reflect this:

> Researcher: I would like to have a conversation with you in regard to your understanding of logarithmic differentiation.

[1] A detailed explanation of these four categories is given in the "findings" section.

P057: Yes sir.

Researcher: I would like to know how you can find $\frac{dy}{dx}$ if $y = (2x - e^{8x})^{\sin 2x}$ or $y = x^{\ln x}$ … Choose any of the two that appear easier for you.

P057: To tell you the truth, personally this is very difficult for me.

Researcher: Okay, so which section of differentiation do you find more challenging among those that you covered in high school and here at the university?

P057: In high school it was a bit fine but here everything is hard.

Researcher: Okay, I know that besides what you did in high school, you were introduced to different rules of differentiation here at the university. Are you able to remember any of the differentiation rules that you have learnt here?

P057: Yes sir, I remember we did chain rule, product rule and other things like differentiating like sine, cosine, and … yes, even logarithms.

Researcher: Okay, that's nice. So, which ones among those rules that you are able to remember can be applicable in any of these two questions I gave you?

P057: To tell you the truth, I love math, but this is difficult for me. It tricks me a lot.

Researcher: Okay, I have seen. Lastly, I would like you to tell me the measures you are putting in place to ensure that you understand these concepts because you will need to teach some of these concepts.

P057: Yes, that's why I found a personal tutor to help me.

Researcher: Is there anything else you feel should be done apart from the help you are receiving from your personal tutor?

P057: I think more tutorials and revisions can help.

Researcher: Okay, you told me you love mathematics, meaning you have a lot of interest in the subject. How do you intend to maintain that interest?

P057: Yeah! I love mathematics and I am trying to find ways of improving. Maybe lecturers also need to take time to explain things to us.

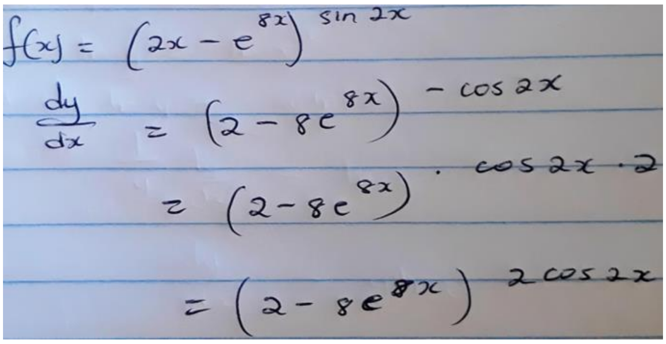

Figure 1. Participant's solution for item 1 reflecting some errors and misconceptions (Source: Field study)

Researcher: Alright, your suggestions are well-noted and thank you so much for your time.

Based on the conversation between the researcher and participant P057, it can be seen that the pre-service teacher is able to remember some differentiation techniques but struggles with applying them to more complex problems, particularly logarithmic differentiation. The participant's proactive approach to seeking help and suggestions for additional support highlight his/her motivation to overcome these challenges and succeed in mathematics.

Action stage

It has been observed that for the 27.8% of responses to item 1 and 28.9% of responses to item 2 that were incorrect, it is likely that student teachers attempted the steps but made significant errors. These students might be performing actions without fully understanding the processes involved, indicating they are operating at the initial action stage. The identified common errors among this group of participants for both items include chain rule misapplication, product rule misuse, and calculation errors in combining derivatives. It was further observed that some of these student teachers could not recognize that solving such problems required the application of logarithmic differentiation. The solution excerpt by a participant identified as P006 as presented in **Figure 1** reflects some of these errors and misconceptions.

Based on the solution presented in **Figure 1**, although this participant was able to differentiate the core function correctly (i.e., $\frac{d}{dx}(2x - e^{8x}) = 2 - 8e^{8x}$), it suffices to mention that failure to recognize the composite nature of the function and the applicability of logarithmic differentiation process meant that success was not guaranteed. A similar pattern was also detected on item 2 as shown in **Figure 2**, a solution by another participant identified as P019.

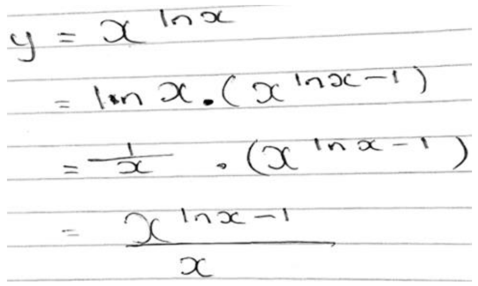

Figure 2. Participant's solution displaying misguided strategies for logarithmic differentiation (Source: Field study)

In the solution presented in **Figure** 2, it can be observed that this student correctly recognizes the application of logarithmic differentiation but does not apply it appropriately to both sides of the equation. A closer look at this solution reveals that the second line $\ln x\,.\,(x)^{\ln x-1}$ is closer to the correct answer, which can be expressed in two forms as $\frac{2\,(\ln x)\,(x)^{\ln x}}{x}$ or $2\ln x\,.\,(x)^{\ln x-1}$. However, the fact that the solution introduces x in the denominator in subsequent steps reveals a student's conceptual misunderstanding of the underlying principles. The student appears to display a misconception of confusing the power rule $\frac{d}{dx}(ax^n) = ax^{n-1}$ with the steps required for logarithmic differentiation, leading to incorrect application of differentiation rules. Similar errors and misconceptions were spotted during the interviews conducted afterwards as evident in the following conversation between the researcher and the participant identified as P064.

> Researcher: … I would like you to explain to me the approach you used (or you could use) to find the derivative of the function $y = x^{ln\,x}$ with respect to x.
>
> P064: Okay sir, allow me to try it first, then I will explain to you.
>
> Researcher: Please go ahead … [after 3 minutes] … Okay, may you kindly explain to me how you approached it?
>
> P064: Yes, whenever we have y like this, we have to write $\frac{dy}{dx}$ then equate to x to the power $ln\,x$. Then the second step is to write $ln\,x$ times x.
>
> Researcher: Okay, then after $ln\,x\ \ times\ x$, what do you do?
>
> P064: Am going to divide by $ln\,x$ on both sides.
>
> Researcher: What do you get after that division?
>
> P064: Am having $\frac{dy}{dx}$ over $ln\,x$.
>
> Researcher: So how do you proceed from there?
>
> P064: That's where I got stuck!
>
> Researcher: Do you remember anything like introducing the natural logarithm to both sides so that you can drop the power and then proceed with finding the derivative?
>
> P064: … yes sir, I remember now! We did this! In this case we could have $ln\,y = ln\,x^{ln\,x}$.
>
> Researcher: Nice! So how do you proceed after that?
>
> P064: We are going to have $ln\,x\,ln\,x$ on the right-hand side. Like I said for the left-hand side, we are going to write $\frac{dy}{dx}$ since we have y there.
>
> Researcher: Okay, so how would you now differentiate $ln\,x\,ln\,x$ on the right-hand side?
>
> P064: I would first write it as $(ln\,x)^2$ since it is multiplication, then drop the power and subtract 1 to remain with $2\,ln\,x$ as my answer.
>
> Researcher: Thank you so much for your feedback. Now tell me, generally do you look at the concept of differentiation as something manageable or something difficult for you?
>
> P064: According to me, it is very complicated.
>
> Researcher: Why do you think this topic is difficult for you?
>
> P064: It requires a lot of mastering because there are too many rules. Sometimes you may apply wrong methods when solving different questions.
>
> Researcher: Alright, thank you very much for your time.

Based on the interview dialogue between the researcher and participant P064, it is evident that the participant did not initially recognize the applicability of logarithmic differentiation for determining the derivative of the given function. Although P064 was able to recall the procedure for logarithmic differentiation after some guidance, this participant subsequently misapplied the power rule without acknowledging the composite nature of the function $\ln y = (\ln x)^2$, which necessitates the use of both the procedure for differentiating logarithmic functions and the application of chain rule.

Process stage

In the partially correct category, which comprises 20.0% of responses for item 1 and 15.6% for item 2, pre-

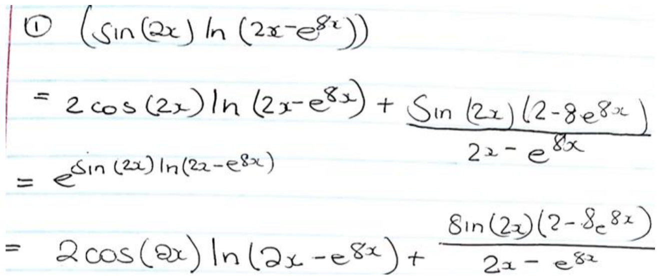

Figure 3. A participant's partially correct solution to item 1 displaying (Source: Field study)

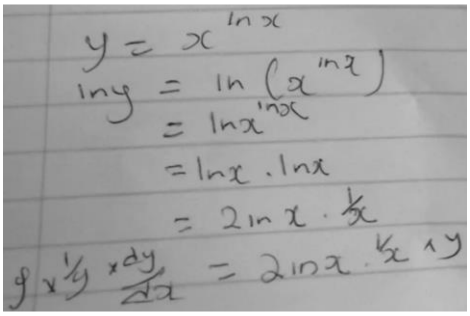

Figure 4. A participant's partially correct solution to item 2 (Source: Field study)

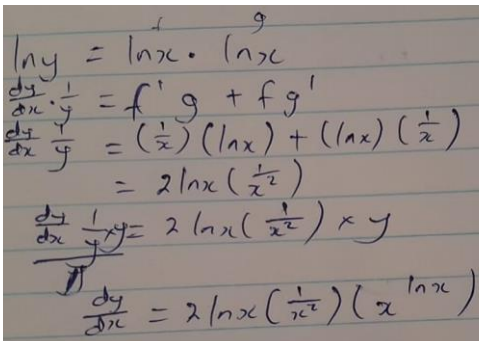

Figure 5. Participant's partially correct solution produced during the interview conversation (Source: Field study)

service teachers display some partial understandings. They could correctly perform some steps or processes but might struggle to integrate all parts correctly. These students are in the process stage in the sense that they have not fully internalized the entire concept into a cohesive object. Typical of this category of responses to item 1 was a situation where after finding the derivatives of individual terms, a student commits errors in combining them using the chain rule and product rule. **Figure 3** illustrates such errors and misconceptions committed by the participant identified as P048.

The excerpt presented in **Figure 3** reflects a participant's partial understanding of the concept of differentiation. In this solution, there are traces of correct recognition of logarithmic differentiation and the composite nature of the function to which both the product and chain rules are applicable. This is particularly evident in the sense that the participant was also able to differentiate both the core function, $\frac{d}{dx}(2x - e^{8x}) = 2 - 8e^{8x}$ and its associated exponent $\frac{d}{dx}(\sin 2x) = 2\cos 2x$ as well as correctly applying the product rule. However, putting these terms together to get a correct solution became challenging due to the participant's failure to apply the logarithmic rule to the left-hand side of the equation. This could be attributed to conceptual misunderstanding, or an oversight where a student fails to recognize that $\frac{d}{dx}\ [f(x)] = \frac{f\prime(x)}{f(x)}$, leading to multiplication by $f(x)$ to both sides. It appears this is the only step that this participant missed to obtain a correct answer.

In regard to item 2, the identified common error was a situation where a student was able to correctly introduce the natural logarithm to both sides and also managed to apply the product rule correctly but forgot to plug in the original value of y. The solution excerpt in **Figure 4** by a participant identified as P013 illustrates this oversight.

The solution excerpt presented in **Figure 4** illustrates a student's demonstration of partially correct solution, which only falls short of substituting $x^{\ln x}$ for y in the final answer. Similar to the solution by P048 in **Figure 3**, this could be attributed to a student's oversight or failure to understand that finding the derivative of this function requires the final answer to be expressed in terms of x. This is why the majority of those who obtained partially correct solutions have been categorized in the process stage as they exhibited an element of no full internalization of the entire concept into a cohesive object. The following interview conversation between the researcher and one of the participants identified as P070 gives further insights into pre-service teachers' partial understanding of the logarithmic differentiation concept. This interview dialogue is based on the solution illustrated in **Figure 5**, which the participant produced during a task-based interview.

> Researcher: Based on the solution you have provided for the derivative of $y = x^{ln\,x}$, kindly explain to me the steps you followed.
>
> P070: After writing $ln\,y = ln\,x\,ln\,x$, I used chain rule to differentiate the right-hand side
>
> Researcher: You say you used chain rule, but I am seeing the product rule in your solution. Are you saying chain rule and product rule are the same?
>
> P070: No sir, they are different. What I did was that I multiplied $\frac{1}{x}$ by $ln\,x$ and $ln\,x$ by $\frac{1}{x}$. Then I added them two and got $2\,ln\,x\left(\frac{1}{x^2}\right)$.
>
> Researcher: I see. Where did you get that x^2 in the denominator? Where is it coming from?
>
> P070: It is coming from $\frac{1}{x}$ plus $\frac{1}{x}$.

Researcher: Remember we are adding and not multiplying … Again, I have seen that you multiplied $2\,ln\,x\left(\frac{1}{x^2}\right)$ by y, which you later replaced by $x^{ln\,x}$. How did that come about?

P070: Because I wanted to remain with $\frac{dy}{dx}$ on the left-hand side, so I multiplied throughout by y. Then y at the end is the same as $x^{ln\,x}$ in the original equation.

Researcher: Oh, perfect! Thank you so much for your time and the solution.

Based on the solution excerpt presented in **Figure 5** and the associated interview dialogue with P070, it is evident that this participant has a good understanding of the overall process. However, there are signs of incomplete internalization regarding the chain rule and product rule. The participants claimed to have used the chain rule when, in fact, they applied the product rule correctly. This suggests a possible oversight. Another evident misconception is that P070 believes $\frac{1}{x}$ plus $\frac{1}{x}$ would lead to $\frac{1}{x^2}$, which is likely an oversight since this error is unexpected at this level of education.

Object-schema stages

Finally, for the correct response category, which includes 44.4% of responses for item 1 and 47.8% for item 2, pre-service teachers had a well-formed understanding of the differentiation process. They could treat the differentiation of logarithmic functions as a single entity and apply their knowledge flexibly, indicating they are at the object or schema stages. This suggests that a sizeable portion of the students have a comprehensive and integrated understanding of logarithmic differentiation. **Figure 6** illustrates solutions to both items by the same participant, identified as P065.

The solutions presented in **Figure** 6 reflect pre-service teachers who demonstrated a comprehensive understanding of the concept. This was further evident during the interview, where the participants falling in this category articulated the procedure clearly and explained all the steps involved using appropriate terminology. The participants' ability to discuss the concept without difficulty indicates a complete internalization of the material, thus reaching the object-schema stage of the APOS theory.

DISCUSSION AND CONCLUSION

Understanding of Logarithmic Differentiation

Overall, results show that 63.9% of the participants demonstrated a substantial understanding of logarithmic differentiation processes, including 46.1% who had reached the object-schema stages. While the study did not specifically measure the instructional impact, the findings still suggest a positive contribution from the ACE teaching cycles that were implemented. These results align with recent studies by Abakah and Brijlall (2024) and Borji et al. (2018), which showed a statistically significant improvement in test scores among learners taught using the ACE teaching cycle. Abakah and Brijlall (2024) further established that this instructional approach significantly enhanced participants' cognitive engagement and development, thereby optimizing their problem-solving skills. Similarly, Koyunkaya and Boz-Yaman (2023) found that the ACE teaching cycle effectively developed students' mental constructions of function transformation through sequential activities.

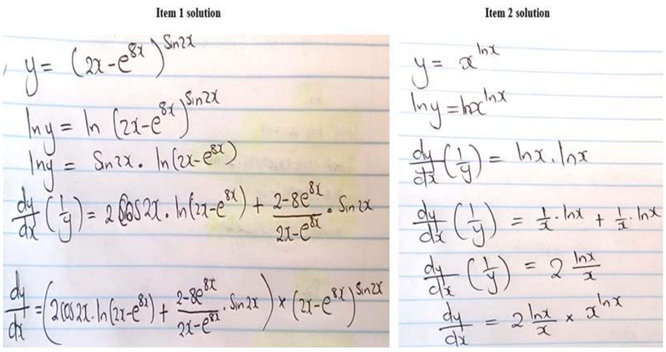

Figure 6. Participant's demonstration of full comprehension of logarithmic differentiation for both items (Source: Field study)

It is also evident that these findings are more promising than those reported in other studies (Feudel & Biehler, 2022; Maharaj, 2013; Moru, 2020), where the majority of students' mental constructions related to the derivative concept primarily centered around actions. However, we are also cognizant of the fact that the present study and previous studies may have focused on different portions of differentiation, and with varying sample sizes. For instance, the current study explored pre-service teachers' conceptualization of logarithmic differentiation, whereas Moru's (2020) study primarily addressed graphical representations of derivatives.

Proficiency in Logarithmic Differentiation

Although a significant proportion of pre-service teachers demonstrated proficiency in logarithmic differentiation, there were still notable challenges, especially in the action and process stages. Common misconceptions and errors were identified among the pre-service teachers. Misapplication of differentiation rules was frequent, with students often misapplying the chain rule and product rule, leading to incorrect solutions. Additionally, some students consistently applied logarithmic differentiation rules incorrectly, failing to correctly introduce the natural logarithm to both sides and incorrectly differentiating the resulting expressions. Regarding misapplication of differentiation rules, both analysis of participants' solutions and interview responses revealed that power rule was at times, used in situations where it was not suitable. This resonates with the findings of a recent study by Mkhatshwa (2024) in which one of the interviewed experienced calculus instructors remarked that "students love power rule so much that they apply it even in contexts where it is not appropriate" (p. 5).

Oversights and calculation errors were common among pre-service teachers, such as failing to substitute the original function back into the final answer and misinterpreting the steps involved in logarithmic differentiation. This suggests that some pre-service teachers struggled to answer questions correctly due to slips[2] or conceptual misunderstandings. This aligns with the observation by Luneta and Makonye (2010) that students' overreliance on procedural knowledge, without a solid conceptual foundation, hinders their ability to generate correct solutions. Similarly, Siyepu (2015) concluded that misconceptions about derivatives of trigonometric functions often obstruct students' success in differential calculus. Othman et al. (2018) also found that students' errors and misconceptions in differentiation were often due to a lack of basic mathematical knowledge, attributing it to students' over-reliance on memorization techniques rather than full utilization of their prior learning.

Implications and Areas of Improvement

Findings from this study shed light on the diverse levels of understanding among pre-service teachers regarding logarithmic differentiation. From a lack of fundamental knowledge to proficient application of differentiation rules, pre-service teachers exhibit a wide spectrum of comprehension. As such we concur with suggestions by Mukuka and Alex (2024a, 2024b) that teacher education curricula should emphasize foundational mathematical concepts that pre-service teachers are likely to teach in schools once they are professionally qualified. Similarly, differentiation techniques are among the concepts taught in schools (DBE, 2011). This could involve targeted courses or modules specifically aimed at improving pre-service teachers' content and pedagogical knowledge. Additionally, ongoing professional development for in-service teachers should address misconceptions related to differentiation rules. Workshops and seminars can provide opportunities for teachers to deepen their understanding and refine their instructional practices.

Practically this study highlights the value of additional tutorial sessions. Institutions should allocate resources for pre-service teachers to engage in focused, small-group sessions where they can clarify doubts and reinforce their understanding of differentiation, and other topics perceived to be difficult. Lecturers should prioritize explaining the underlying concepts behind differentiation rules, as understanding the "why" alongside the "how" can prevent inappropriate application of rules. Integrating tools like Desmos and GeoGebra into instruction as suggested by Mkhatshwa (2024), can also enhance visualization and conceptual understanding. These platforms allow students to explore graphs and functions dynamically, reinforcing their comprehension of derivatives.

Theoretically, our findings align with Maciejewski's (2023) work on procedural flexibility and highlight the importance of ACE teaching cycles for enhancing pre-service teachers' understanding of logarithmic differentiation, and derivatives in general. It is worth noting that ACE cycles integrate active learning, reflection, and application, potentially enhancing conceptual grasp (Arnon et al., 2014). According to Maciejewski (2023), encouraging students to calculate derivatives using multiple methods fosters a deeper understanding of the underlying principles. Siyepu (2013) emphasizes the importance of highlighting differentiation rules and the specific contexts to which they are applicable. Explicitly addressing cases where certain rules do not apply can prevent common errors arising from over-generalization.

It is suggested that future research may focus on investigating how pre-service teachers transfer their knowledge of differentiation rules to various problem-solving scenarios. There is also a need to explore the effectiveness of ACE teaching cycles in scaffolding pre-service teachers' understanding of derivatives.

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[2] According to Olivier (1989) cited in Siyepu (2013, p. 578), "Slips are wrong answers owing to processing; they are not systematic, but are carelessly made by both experts and novices; they are easily detected and are quickly corrected."

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