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Undergraduate students' understanding of the application of integral calculus in kinematics

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Abstract

In calculus, students can integrate functions that require procedures or algorithmic rules, but they grapple with contextual problems involving real-life motion of physical bodies. When undergraduate students learn the application of integration, they are expected to comprehend the concept of integration and apply it to optimization. This study used the action-process-object-schema (APOS) theory to determine undergraduate students' construction of the application of integral calculus to kinematics. This study was qualitative and involved a case study of 150 secondary mathematics students registered for a Bachelor of Education degree at a university in South Africa. Data were collected through a written test by all the students and semi-structured interviews with eight students. The eight students were selected purposively, and the interview questions were meant to clarify some of the responses raised in the test. The content analysis of the written responses was done to reveal the stages of students' concept development of kinematics. The findings revealed that students had significant challenges performing second- and third-level integration. These involve substituting the initial conditions at least once to find the constant integration for each level. Furthermore, students' connection with displacement, velocity and acceleration concepts was weak, coupled with their failure to consider the point when the object was momentarily at rest.

Keywords: application of integration, kinematics, APOS theory, schema, undergraduate students

INTRODUCTION

Calculus is a branch of mathematics that is concerned with limits, differentiation and integration of functions. Undergraduate mathematics education students encounter integral calculus in their second year of study in South Africa. This comes after having done limits and differential calculus in the last year of high school and first-year undergraduate. Integration focuses on the determination of the anti-derivatives and their application to functions of a diverse nature. Determination of integrals is achieved conceptually by means of definite integrals (as area under the graph and Riemann sums) or as a process by means of indefinite integrals (calculating anti-derivatives) (Hogstad & Isabwe, 2017). Definite and indefinite integrals are essential in calculus as they form the basis for many real-life problems commonly encountered in physics, engineering, biology and economics. The contextual problems constitute the application of integration, which comprise volumes of revolutions, centroids, rates of change, moments of inertia, pressure, total costs and so on (Ferrer, 2016).

To understand integration, students need to have pre-requisite topics like trigonometry, algebra, analytic geometry, limits, differentiation and continuity. Integral calculus is a gateway to advanced mathematics courses such as vector calculus, differential equations, multivariate calculus, complex and real analysis. In calculus, students frequently encounter integration more than limits or differentiation (Jones, 2013). When undergraduate students learn the application of integration, they are expected to comprehend the concept of integration and make connections between particular constructs within integration. As integration is a continuance of differential calculus which was covered in the previous year, the focus was on instantaneous rates of change which require students to

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Contribution to the literature

- This study revealed the stages of students' concept development in the application of integration to kinematics and identified the associated errors at each stage.
- The findings demonstrated that students had significant challenges in performing second- and third-level integration, which involved substituting the initial conditions at least once to find the constant of integration for each level.
- The students' understanding of displacement, velocity and acceleration concepts was weak, coupled with their failure to consider the point when the object was momentarily at rest.

integrate functions and plug in boundaries conditions to find displacement or velocity. In the application of derivatives, students were differentiating to find acceleration or velocity. While physics majors may resort to equations of motion to solve kinematics problems, mathematics majors are required to construct knowledge in differentiating and integrating given functions to obtain unknown quantities in motion in one dimension.

Integration is a valuable topic which serves as the basis for diverse real-world applications (Jones, 2013). In learning integral calculus, undergraduate students are expected to construct mental structures that enable them to solve problems in the application of integral calculus to motion in one dimension. The organized structure of knowledge is what Skemp (1962) termed schema, and to understand something means to assimilate it into an appropriate schema. In integral calculus, students are exposed to the rules, procedures and algorithmic formula for standard functions, for example, the u-substitution, integration by parts, the power rule, partial fractions, and logarithmic and trigonometric substitutions. Students might be fluent in performing symbolic techniques of integration, but may lack the conceptual understanding thereof, which becomes a stumbling block when they try to apply integration to real-world problems (Nguyen & Rebello, 2011). Thompson and Harel (2021) corroborate that students can evaluate integrals but cannot relate their computations to other contexts.

Students grappling with contextual problems involving the real-life motion of physical bodies stems from their limited knowledge of the concept of rates of change (Kouropatov & Dreyfus, 2014). Students' challenges in kinematics are related to the usage of formulas and understanding the problem. Kinematics is defined as the relationship between the position of a particle and its motion without considering the forces acting on the particle (Taqwa et al., 2022). Very few studies report on specific difficulties that constrain undergraduate students' success when solving kinematics problems on integration (Zulu et al., 2021). Therefore, this study sought to explore how undergraduate students constructed their knowledge and understanding of the application of integral calculus in motion in one dimension. The research question was. "What are undergraduate students' understanding of the application of integral calculus in kinematics?" In other words, what mental structures do students operate when they conceptualize the application of integral calculus to kinematics?

LITERATURE REVIEW

Integral calculus has been an object of study by mathematics educators for a long time at the undergraduate level. A thorough search revealed that most studies in integral calculus are conducted at the undergraduate level in most countries. The exception was two studies by Kouropatov and Dreyfus (2013, 2014), which in either case investigated high school students' construction of the concept of integration as a totality of small parts. The first category of studies on integral calculus involved a quest for using theory to understand students' mastery of the techniques of integration. A study by Brijlall and Ndlazi (2019) used activity sheets and follow-up interviews to explore 30 first-year engineering students' understanding of techniques of integration in South Africa. The study was structured on the action-process-object-schema (APOS) theory. The findings established that students exhibited procedural tendencies in evaluating integrals, but their conceptual understanding of integrals was shaky. The students could not define definite and indefinite integrals. In another study, Ndlazi and Brijlall (2018) sought to explore engineering students' understanding of the techniques of integration by using the three worlds of mathematics framework. Tasks on integration were administered and followed by open-ended interviews. The findings suggested that students only attained the lowest of the three worlds, namely, the conceptual/embodied. The previous studies relate to this study in the sense of using established theories to investigate undergraduate students' conceptualization of integration. This study goes further by investigating students' understanding of the application of integration, in addition to the techniques of integration.

In another study on the techniques of integration by Ferrer (2016), students encountered challenges with certain functions, for example, trigonometric functions. The students' difficulties in evaluating integrals were attributed to these challenges. After analyzing students' responses to an examination and probing in open-ended interviews, the findings showed that students' lack of basic mathematics knowledge muffled their attainment in integration. Kiat (2005) posited that structural errors were the largest, which signify students' lack of specific mathematics content. The study by Kiat (2005) also indicated that students had difficulties with questions that involved trigonometric functions, and students seemed to focus more on the procedural aspects of integration rather than the conceptual. Insights from these studies informed this study to investigate undergraduate students' challenges in grasping the techniques of integration as they unfold in the application of integration.

Social sciences students' concept images and definitions of anti-derivatives were the objective of a study by Moru and Qhobela (2019) in Lesotho. In that study, data were collected through responses to a task-sheet on evaluating anti-derivatives given to students. Interviews were also conducted to further clarify students' written responses and the concept image and concept definition theory was used as lens to analyze data. The findings confirmed that students' concept images of anti-derivatives in the integration of algebraic expressions were coherent, but some students still had challenges with the integration method to use in solving problems.

Similarly, Serhan (2015) conducted a study to investigate 25 undergraduate calculus students' procedural and conceptual knowledge in definite integrals. After administering a test, written responses highlighted that students were dominant in procedural knowledge relative to conceptual knowledge. Their conceptual understanding of definite integrals was limited, and they could not represent the concept in other contexts. Thus, literature indicates that integration is a concern to mathematics educators as students grapple with it.

The second category of studies in integral calculus focuses on students' understanding of the application of integration into kinematics. Frequently, students master the appropriate methods of integration but fail to establish a connection to apply that knowledge to real-world problems as commonly found in engineering and sciences. Jones (2015) asserts that students' understanding of definite integrals was insufficient to help them make sense of contextualized integrals. In that study, students had a clear understanding of the area under the graph as a definite integral. However, this could not help much since many applications in integration do not make use of this property. In light of this, this study takes a furtherance of the quest to find a connection between knowledge of integration and its application in contextual problems.

Some of the studies on kinematics were conducted with physics students since it is one of their topics too. Nevertheless, Nguyen and Rebello (2011) sought how physics students understood and applied the area-under-the-graph concept in solving introductory physics problems. Very few students could realize that the concept was applicable to physics problems and in some cases, it could not be applied to novel situations. Sundari et al. (2023) probed students' understanding of kinematics in a study wherein a survey of 129 physics students was conducted. After responding to a written test, it was discovered that students struggled to solve problems in kinematics, especially those that required identifying motion through a graph. In a similar study, Taqwa et al. (2022) inquired into the effectiveness of using diagram-based motion to improve students' conceptual understanding of kinematics. After administering a multiple-choice task to an experimental group of 36 physics students, Taqwa et al. (2022) showed that the use of diagrams can enhance conceptual understanding of one-dimensional motion.

However, many students remained with difficulties in determining instantaneous velocity based on the displacement-time graph. In kinematics, many assumptions need to be formulated in the process of learning displacement, velocity and acceleration. This pertains to the direction of motion and boundary values. Taqwa and Rivaldo (2018) established that a sample of 48 physics students harbored wrong assumptions regarding the displacement function $x(t)$ and assumed that the negative value of acceleration denotes the particle is slowing down. The studies above have strong implications for the current study since about a third of the participants in this study were physics education majors. These students cover kinematics in both physics and mathematics.

Under the APOS theory, the goal of instructors is to assist students in attaining higher levels of mental structures in understanding a mathematical concept. Undergraduate students in a study by Maharaj (2014) only attained the action conception which saw them being able to perform step-by-step integration when the integration is explicit. However, they had difficulties applying the rules of integration and the method to use is implied due to unattained higher levels of process and object mental structures.

This literature review overview points to the following: integral calculus is dominantly offered in undergraduate studies; written tasks in the form of tests, tasks and examinations are used determine the level of students' conceptualization of integration and its application; open-ended interviews are also commonly used as a follow-up to written tasks to clarify further the written responses; analysis of data is done through the lenses of mathematics-related theories; and exploring undergraduate students levels of attainment in mathematical concepts informs the development of appropriate didactics when teaching integration in higher education.

THEORY

In research, a theoretical framework essentially gives structure to a study and is useful in data analysis. To understand how students develop knowledge of mathematical concepts, the APOS theory was used. APOS theory was developed by Dubinsky in 1985 (Arnon et al., 2014). It was developed as part of an effort to understand how mathematics is learned. It tries to understand how students construct different mathematical concepts and suggests pedagogic actions that can stimulate the learning process. According to APOS theory, an individual deals with a mathematical situation by using certain mental mechanisms to build cognitive structures that are applied to the problem situation (Dubinsky et al., 2005). Mental construction in the APOS framework implies that there are differences in everyone's description of each stage. This can be influenced by the thinking and ability of each individual. The novelty of this framework lies in the students' mental constructions through detailed explanations at each stage in the APOS framework on the application of integration concepts. Hence, this study describes students' mental construction of the application of integration concepts based on differences in mathematical ability. This research contributes to guidance for educators in improving students' understanding.

An APOS inquiry starts with the researcher performing a theoretical analysis of the mathematical concept concerned using his expertise as a mathematics education researcher and instructor. The purpose of the theoretical analysis of a concept is to propose a description of specific mental constructions that students might require to develop their understanding of the concept. This theoretical analysis is also called genetic decomposition (GD). An action is an externally motivated transformation of previously conceived objects. Each step of the transformation must be performed overtly and guided by external and ordered instructions (Arnon et al., 2014). An action forms the crucial beginning of understanding a mathematical idea. Hence, this learning-theory-based pedagogical approach begins with activities designed to help students construct action conceptions. Actions transform into processes when they are reflected upon and repeated. An individual moves from relying on external cues to having internal control over transformations. At this level, students can work in reverse and skip some steps or perform the steps mentally. When a student encapsulates a process, an object is formed in the mind of the student. At the object level, the student's focus moves away from the mathematical concept as a dynamic transformation into a static entity upon which further actions and processes can take place. Finally, a schema is an individual's coherent collection of actions, processes, objects, and other related schemas which form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept (Asiala et al., 1996). While this structure describes how an individual may construct a single transformation, a mathematical concept often involves many actions, processes and objects that need to be organized and linked into a coherent framework, which is called a schema. It is coherent in the sense that it provides an individual with a way of deciding which mental structures to use in dealing with a mathematical problem situation.

A GD was used in this study to develop pedagogical knowledge and settings. The GD also explains the students' expected performances that indicate differences in the development of their constructions in a specific concept. The GD for the application of integration in kinematics is shown in **Table 1**.

Table 1. The genetic decomposition of the application of integration in kinematics

Mental structure	Constructs in kinematics
Action	Given any function, an individual is expected to evaluate the indefinite using one of the integration techniques. The individual does every step learnt in class (Brijlall & Ndlazi, 2013) in the solution process of a given mathematical problem.
Process	Interiorize indefinite integrals to definite integrals where integration is the area under the curve in $units^2$. An individual can integrate transcendental functions where the area goes up to infinity where applicable. An individual can find the function whose derivative yields given integrands by working in reverse, guided by the Fundamental Theorem of Calculus. Students perceive the application of integration to kinematics, solids of revolution, centroids and others by applying limits of integration to get output.
Object	Perceives kinematics in totality and considers the direction of motion to find unknown quantities. Given acceleration, an individual can perform repeated integrations to evaluate velocity and displacement by using appropriate initial and final conditions. The individual carries out action/process transformations to find time and velocity when the particle is momentarily at rest. An individual realizes that the area under the acceleration-time graph is velocity and the area under the velocity-time graph is displacement.
Schema	Given any problem, an individual can identify the appropriate application and solve the problem using the correct technique of integration. An individual considers the assumptions of kinematics to solve a given problem.

METHODOLOGY

This qualitative study was conducted at a public university in South Africa as an explorative case study design. A case of 150 second-year undergraduate students studying towards an education degree participated in the study. In APOS theory, the next phase after the theoretical analysis and development of GD is the implementation of instruction in accordance with the hypothesis in the GD. The implementation of instruction provides an opportunity for the collection and analysis of data, which is carried out using the theoretical lens of APOS theory (Asiala et al., 1996). The concept of integration was taught using traditional instruction, and APOS theory was used as an analytical evaluative tool in this study (Dubinsky et al., 2013). Only one problem with three items was selected from the end-of-unit test that was written by all the students. The item was used to identify and analyze the students' understanding of kinematics at all stages of APOS theory.

The problem under consideration was, as follows:

A particle P is moving on the x-axis and its acceleration $a\ m/s^2$, t seconds after a given instant, is given by $a(t) = 8 - 2t$ for $t \geq 0$. Initially, P is on the positive x-axis 84 m away from the origin O, and is moving towards O with a speed of $7\ m/s$.

(1) *Find an expression for the velocity of P.*

(2) *Determine the times when P is instantaneously at rest.*

(3) *Show that when $t = 12$, P is passing through O.*

Action conception is required to execute step-by-step integration of the polynomial two times. In some cases, students performed the integration mentally or skipped some steps as evidence of their process skills. Further process skills were manifested when students used the initial conditions to find the value of the constant of integration for both the $v(t)$ and $x(t)$ expressions. The object conception in this item centers on the need to assign signs based on the direction of the particle and show that displacement is zero at $t = 12$. Students applied action/process skills to this object by substituting $t = 12$ in the just-obtained displacement expression. The test item was a fairly standard question on kinematics, and they were analyzed in traditional grading ways. The responses were graded on appropriate scales from incorrect to correct with partial crediting in-between. The frequencies of the scores were captured.

The written responses were used to design semi-structured interview questions where students might be asked to explain what was written or whether they wished to revise any of the responses (Arnon et al., 2014). A group of eight students was purposively selected for the follow-up interviews; three were from the wrong responses category (K15, K65, and K98), three from the partially correct responses (K3, K38, and K148) and two from the correct responses category (K3 and K19). The interviews were held to clarify further the selected students' responses and the possible reasons for those responses (Maharaj, 2014). The interviews were audio-tapped and transcribed by the researcher and the transcripts complemented the students' written responses from the test. The written responses and transcriptions were analyzed by categorizing the data into themes and the classification of errors to explore undergraduate students' understanding of the application of integration in kinematics. Orton (1983) identified three types of student errors, namely, structural, executive and arbitrary. The structural error arises from the failure of a student to grasp the principle necessary for the solution. Kiat (2005) regarded this error as conceptual. An executive error (or a procedural error) is one which involves failure to carry out algorithms despite having understood the concept involved in the problem. Finally, an arbitrary error, also called a technical error, refers to an error due to carelessness or failure to consider the constraints laid down in the question.

Table 2. The frequencies of students' constructions of knowledge in the test

Type of response/item	(1)	(2)	(3)	Total
Blank (B)	18	25	62	105
Wrong (W)	44	99	64	207
Partially correct (P)	77	15	22	114
Correct (C)	11	11	2	24
Total	150	150	150	450

FINDINGS

After data collection, analysis of data was done through the lens of the APOS theory to ascertain those students attained the mental structures called for in the GD and how well they achieved them (Arnon et al., 2014). The frequencies of students' performance are displayed in **Table 2**.

Of concern is the record of students who did not provide responses to the three items, which was about 23%. This task was part of the students' formative assessment; hence, there was no obvious reason students had to skip the questions other than the difficulties in understanding the concept of kinematics. For item (3), the skip rate was higher at about 41%. The sum of incorrect responses provided by the students stood at 207, which was relatively high for all three items. In item (2) only, much more than half of the students provided incorrect responses. Partially correct answers were most common for item (1) relative to the other types of responses. Out of the combined responses of 450, only 24 were correct, representing 5%. Item (3) was the least performed, with only two students providing perfectly correct responses. The performance of the students was negatively skewed, a testament to students' difficulties in the application of integral calculus to kinematics.

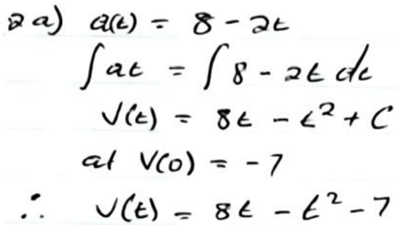

Figure 1. Correct solution to item (1) by K19 (Source: Authors' own elaboration)

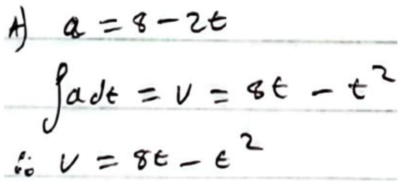

Figure 2. Absence of a constant of integration by K38 (Source: Authors' own elaboration)

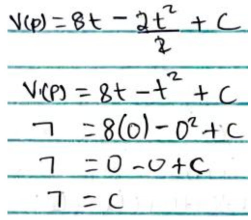

Figure 3. Solution with c = 7 by K95 (Source: Authors' own elaboration)

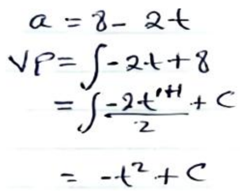

Figure 4. An incorrect attempt to integrate (Source: Authors' own elaboration)

Item (1) Analysis

Only eleven students successfully solved this item, which required students to integrate the polynomial once and find the constant of integration using the given initial conditions. **Figure 1** illustrates the correct solution.

The integration of the given polynomial was standard, but the initial conditions needed careful consideration. Participants like K19 managed both the integration and initial speed. Other than these eleven, the other students encountered challenges in their attempts. The P-category participants had some evidence of attempts to integrate the polynomial, but the process was flawed. Of the 77 students in the category of P, 40 integrated the polynomial but did not include the constant of integration. As a result, they had no use of the initial condition of a velocity of 7 meters per second (illustrated in **Figure 2**).

A follow-up interview with K38 revealed that it was a procedural error. R stands for the researcher in the dialogue.

> R: What do you get when you integrate acceleration?
>
> K38: I get velocity, as you can see there.
>
> R: Do you think the integration on the right-hand side is correct?
>
> K38: Yes. I don't see any problem with it.
>
> R: Have you ever heard of a constant of integration, and under what circumstances do you apply it?
>
> K38: Whenever we perform indefinite integration. Which was not the case here, Sir.
>
> R: Are you sure?
>
> K38: Oh, I see. Even indefinite integrals do have a constant initial, which can be evaluated using initial conditions. I am supposed to have a c.

K38 was also unsure of indefinite integrals in relation to application problems. The constant of integration is part of the procedure, which is replaced by the initial values in the case of application problems. A further eight students integrated and appended the constant of integration c but then did not go on to evaluate it. They did not use the given initial velocity. Twenty-one students attempted to evaluate the c but obtained 7 instead of the expected -7. **Figure 3** illustrates K95's response.

The remaining eight participants in this category did the correct integration of the polynomial but also made mistakes on the value of c, whereby diverse values of c were obtained. For instance, three students got $c = 84$ after using the initial condition of 84 meters. They disregarded the fact that 84 meters was the initial displacement of the particle, which was used in a velocity function.

In the W category, four students tried to integrate the acceleration function to get the velocity function, but they failed to do so. **Figure 4** illustrates K26's response which shows flawed integration. The constant value 8 was dropped.

In the same category, K149 integrated and obtained $v(t) = 8 - t^2 + c$. Thus, after substituting the initial velocity of 7 m/s, the final velocity function was $v(t) = 7 - t^2$. Two participants, K44 and K50, attempted to integrate but used the formula for arc length and volumes of revolutions, respectively. The rest of the students in this category did not apply the idea of integration in their attempts to solve the problem. Three students utilized the equations of motion to find velocity

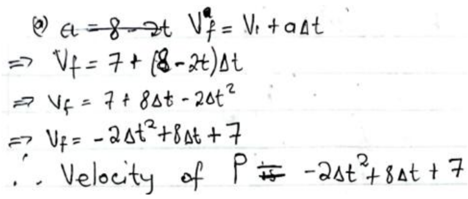

Figure 5. Use of the equations of motion (Source: Authors' own elaboration)

functions. For example, K19 incorporated $v_f = v_i + a\Delta t$ while K98 used $v = 2a + at$ but their resultant functions were incorrect. The latter formula is unknown in the equations of motion found in physics.

Figure 5 illustrates K19's response.

To find out more about the application of equations of motion, an interview was arranged with K98.

R: Why did you use the equations of motion?

K98: Because I want to find the final speed.

R: Ok, but where did you use the idea of optimization?

K98: I did not because we do it this way in physics.

R: What comes to your mind when acceleration is given as a function?

K98: I-I

R: Optimization translates to the point when the object is momentarily at rest.

K98: But this works for us in physics.

R: It is only possible if acceleration is constant, but it is linear here.

K98: Thank you, Sir. I never saw it that.

The dialogue shows that equations of motion are not applicable for instantaneous acceleration and velocity.

Instead of integrating, some students chose to differentiate the acceleration function to the expression for velocity. To some students, velocity is the first derivative, without realizing that it is also the first integral of acceleration. K15 responded this way: $v = a'$ then $a' = -2$. In the interview, K15 responded that he was not sure whether he had to integrate or differentiate. K133 adopted the idea of differentiation but did not actually perform it (shown in **Figure 6**).

Some students had a mix-up of the displacement, velocity and acceleration concepts. K62 had an idea that it is the displacement function which is supposed to be differentiated. Hence, he applied $v = \frac{\Delta x}{dt}$. This was simplified further to $7 = \frac{84}{t}$, giving rise to $t = 12$ seconds. However, the question required the velocity function.

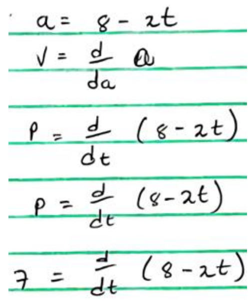

Figure 6. Acceleration is conceived as a derivative of velocity (Source: Authors' own elaboration)

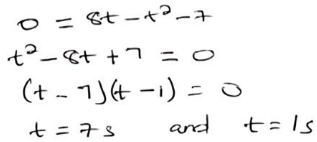

Figure 7. Correct values of t when the particle is at rest (Source: Authors' own elaboration)

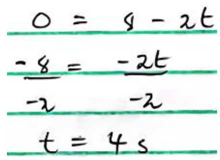

Figure 8. Equating acceleration to zero by K133 (Source: Authors' own elaboration)

Similarly, students like K4 had the dilemma of ending up with the value of time instead of the velocity by solving the equation involving velocity: $7 = 8 - 2t$. K38 also solved this way.

Item (2) Analysis

In this item, students were expected to note that when the particle was momentarily at rest, then the velocity was zero. Having obtained the correct expression for velocity in part (1), only eleven of them also went ahead and computed the correct values for t when $v = 0$. **Figure 7** illustrates a correct solution by K3.

However, 26 students thought that momentarily at rest meant the time when acceleration was zero. They tried to find t by solving the $8 - 2t = 0$, as shown in **Figure 8**.

Another five students solved for t by saying $8 - 2t = 7$. This was a mix-up because acceleration on one side cannot equal initial velocity to the other. The dialogue with K3 is shown below:

R: What does *momentarily at rest* mean?

K3: Acceleration is zero.

R: Are you sure?

K3: Oh no. We say $v = 0$. I made a mistake.

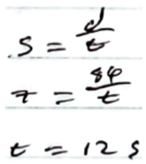

Figure 9. Using displacement and velocity to find time by K38 (Source: Authors' own elaboration)

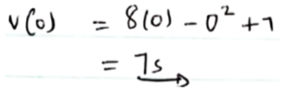

Figure 10. Evaluating $v(0)$ instead of $v(t) = 0$ by K55 (Source: Authors' own elaboration)

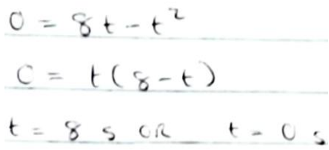

Figure 11. Use of incorrect expression for velocity by K29 (Source: Authors' own elaboration)

Nine participants thought of using displacement and velocity to find time. After solving $7 = \frac{84}{t}$, they got $t = 12$, as shown in **Figure 9**. However, no optimization is involved if solved this way. It is average velocity which they calculated. Students like K38 disregarded the fact that when the particle was at rest, $v(t) = 0$. In the same line of reasoning, three participants evaluated $v(0)$ instead of $v(t) = 0$, as shown in **Figure 10**.

The value of the constant in $v(t)$ is also incorrect. When asked why he calculated the value of initial velocity $v(0)$, he said he thought $v(0)$ is the required solution when the particle is *momentarily at rest*.

Finally, some students thought of using the numbers given in the question in one way or another but without understanding. For example, six students computed $a(12)$ but then used the acceleration function as: $a(12) = 2 - 8(12)$. However, 12 seconds was meant for the item (iii) on displacement. Also, two students tried to solve the equation $x(t) = 0$ to find the time when the particle was at rest. The equation $x(t)$ was obtained by first integrating the velocity function. However, that complicated the problem because it meant solving a cubic equation.

About 20 participants faltered in finding the accurate time values simply because the velocity expression from (1) was incorrect. They equated the faulty expression for velocity appropriately to zero as illustrated in **Figure 11**. The velocity expression only lacked constant integration.

The participants would have obtained the correct solutions if not for the faulty equation. Solving quadratic equations is something which second-year students can easily do. Similarly, some students had difficulties solving the equation because they had obtained other values of the constant of integration other than $c = -7$.

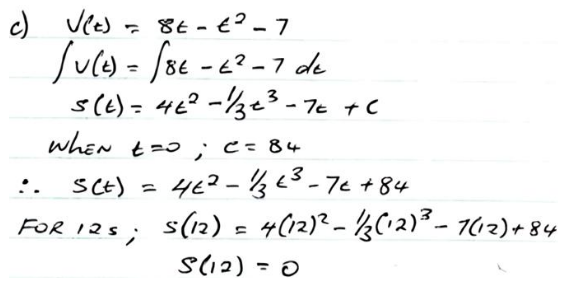

Figure 12. The expected solution in (3) (Source: Authors' own elaboration)

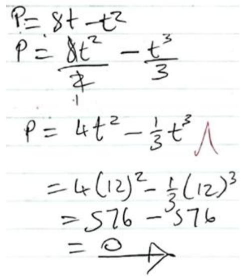

Figure 13. An otherwise correct integration but lacking the constant of integration by K49 (Source: Authors' own elaboration)

Thirteen students used $8t - t^2 + 7 = 0$, but then they had difficulties solving it since it had no factors. Some had $c = 55$ and $c = 84$. As a result, the correct values for time could not be found. Having determined the correct equation for $v(t)$, six students were unsuccessful at solving the ensuing equation $8t - t^2 - 7 = 0$. It was a matter of failing to solve the quadratic equation, which is often addressed in junior high school. Upon inquiry in the interview, K148 responded that she got mixed up with the negative signs but could solve any quadratic with or without integer solutions.

Item (3) Analysis

The participants' performance in this item was not good. Firstly, the frequency of students who skipped this item was at least 41%. These students made no attempts to answer this item. On the other hand, there were only two participants who scored fully on this subject. K19 correctly does the expected solution in **Figure 12**.

The students were expected to integrate the velocity function from (1), consider the initial conditions and then show that $x(12)$ was equal to zero. Displacement was zero meters because the particle moved towards its origin. Fifteen participants attempted to integrate the function but did not include the constant of integration in the $v(t)$ expression, as shown in **Figure 13**.

The error of omitting the constant of integration happened more than once in K49. Six students integrated the function and got $k = 84$ but did not go ahead to find $x(12)$, as shown in **Figure 14**.

In another instance, one participant performed the integration perfectly but failed to evaluate the constant

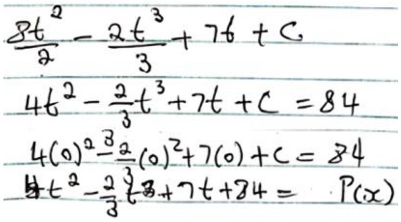

Figure 14. Incomplete solution lacking the value of $x(12)$ (Source: Authors' own elaboration)

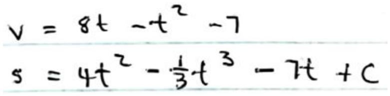

Figure 15. A correct integrand but without the value of c by K45 (Source: Authors' own elaboration)

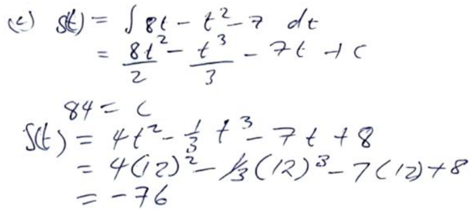

Figure 16. An arbitrary error spoiling the final part of the solution (Source: Authors' own elaboration)

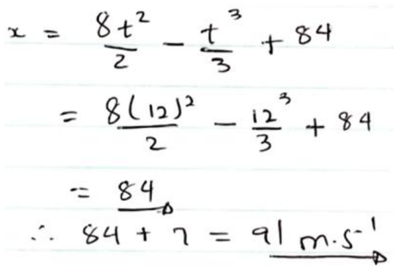

Figure 17. Failure to fully integrate the polynomial correctly by K55 (Source: Authors' own elaboration)

of integration; hence, evaluating $x(12)$ could not be possible (as shown in **Figure 15**).

A further four participants integrated $v(t)$ well but then got the wrong value for the constant of integration. Consequently, verifying that displacement is zero when $t = 12$ seconds could not be attained. To some students, it was a carry-over error emanating from the wrong constant of integration for the velocity expression. This was unavoidable, and the final answer for $x(12)$ could not be determined.

Arbitrary errors led to incorrect final answers for some students. K14 was on track until he wrote $k = 8$ instead of the just-obtained $k = 84$; hence, she could not confirm $x(12) = 0$ (shown in **Figure 16**).

Participants in the category of Wrong answers either did not perform integration, or if they did, they carried it out wrongly. At least 43% of the students got this item wrong. Eight participants failed to integrate the polynomial of velocity to get displacement, as shown in **Figure 17**. This was a technical error, as they all were

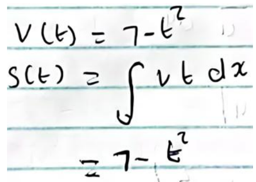

Figure 18. No attempts to integrate (Source: Authors' own elaboration)

expected to know how to integrate the quadratic polynomial.

The expression of $x(t)$ by K55 is missing a constant term; hence, his $x(12)$ could not yield a zero as expected. This was a result of the wrong integration process of the polynomial. Five participants did not integrate the polynomial despite attempts to do so. K149's work illustrates this point in **Figure 18**.

Another case of failure to integrate was seen in the work of K26, K137, K105, and K105, who computed displacement at $t = 12$ seconds by substituting into the acceleration expression. Four other students also solved the problem this way. Thus, they all obtained $P = 8 - 2(12) = -16$ meters. Lack of integration was also seen in the work of eleven students who calculated time using $\frac{84}{7}$, which had no connection to the demands of the question. Three students used the equations of motion but to no avail. Some participants also considered $\frac{84}{12}$, but it does not apply when displacement is given as a function. K115 applied $\Delta x = v_i \Delta t + \frac{1}{2} a \Delta t^2$ while the rest used $v = u + at$. At the least, the former is related to displacement, while the latter is not. Finally, K65 decided to adopt the formula for the volume of revolution and wrote $\pi \int_0^{84} (8 - 2(12))^2$. Upon further inquiry, K65 replied that he was not sure of the question and, at the same time, he did not want to leave the item unanswered. Thus, he came up with the idea of volumes of revolution, which he also doubted. More so, the integrand is a constant value with a dubious origin.

DISCUSSION AND CONCLUSION

Based on the frequencies in **Table 1**, about 23% of the students operated at the pre-action stage in all the items (Kazunga & Bansilal, 2017). Santos (2019) asserts that students' level of mathematical performance correlates with the levels of understanding they attain as they solve a task. Even the APOS theory provides a framework to organize students' thinking as a mathematical concept (Langi et al., 2023). Hence, without attempting to solve a problem, categorizing students' knowledge construction in a particular concept is inapplicable. The blank responses cannot even be classified under the error-types of Orton (1983). For that to be possible, at least some sort of attempt must be made.

Teaching and research results have shown that developing an object conception for a concept is difficult and takes a long time (Arnon et al., 2014; Asiala et al., 1998; Trigueros & Martínez-Planell, 2010). Hence, results showed that for all the three items, only 5% attained the object level. Students in this study could not adequately go beyond performing procedures and explaining the steps required to accomplish the application of integration. According to Borji and Font (2019, p. 13), students should be able to "know the reason for using any technique and concept and know where, and when, and for what they have to use each of them". Only two students managed to reason and justify the causes for the steps they took to evaluate the displacement. According to the GD, this mental construction represents the object conception of the application of integration to kinematics.

Besides the students who skipped the questions and solved the kinematics problem correctly, 72% committed diverse errors in their attempts. Thus, it seemed appropriate to classify the types of errors made by students using Orton (1983). The integration of polynomials is straightforward and requires action and process skills mainly. However, most students did not attain that. Structural errors were observed in students' responses categorized as wrong for both (1) and (3), whereby no attempts to integrate the polynomials were seen. For example, some students used equations of motion, while others differentiated instead of integrating. It happens that students often confuse integration with differentiation, as the former also relies on derivatives (Kiat, 2005). Ferrer (2016) posits that students confuse and interchange the formulae for integration and differentiation only when transcendental functions are involved. Hence, Taqwa et al. (2022) note that velocity is difficult for students to understand. Depending on what is provided first, velocity can be a result of differentiation or integration if given displacement and acceleration, respectively. In cases where integration was contemplated, other formulae were applied, for example, arc length and solids of revolution. In problems that contain some numbers, students sometimes try to use the formulas which come to their minds (Taqwa et al., 2021). Under structural errors, students' difficulties were in recognizing that integration was needed to solve for velocity and differentiation (Khan et al., 2012). Kiat (2005) reports that students are better at integrating polynomial functions relative to transcendental functions.

In calculus, integration has a dual nature, as an antiderivative requiring the use of integration techniques and as a calculation tool in application problems (Maharaj, 2014). This compound nature of integration led to a proliferation of executive errors for the students who were in the partially-correct category. The greatest challenge was omitting the constant integration in both the velocity and displacement expressions and their proportions were 40% and 17%, respectively. Langi et al. (2023) reveal that often, students do not know the meaning of the constants they write on the integration results. This challenge represented an executive error whereby students understood the rules for integrating polynomials but failed to carry out all the steps (Maharaj, 2014). Moru and Qhobela (2019) say a procedural error is committed if a constant of integration is omitted on the one hand, while there is a possibility of a technical error if the students just forget to write the c on the other hand. It also meant the action and process skills were present but not robust. Despite making some attempts to include the constant of integration, 16 students did not evaluate it, and those who did attempt got it wrong.

In item (2), nearly half of the students failed to equate the velocity expression to zero, which was evidence of structural errors. Students were used to finding moments when the particle was at rest by using $x'(t) = 0$ but this time it was $\int a(t)dt$. In this category, some solved for $a(t) = 0$, thus using the given acceleration function while two of them chose $x(t) = 0$. These two instances blur the distinction between structural and executive error types; it is structural because velocity was not involved, but executive because students reckoned that an expression is equated to zero at the point the particle is at rest. Under executive errors, students were hampered from solving the homogenous quadratic equation mainly due to errors incurred in (1) concerning the constant of integration, which affected about 20% of the students. In such a case, the quadratic expression could not be factorized, and some students doubted the non-integer solutions they got. The students' action and/or process skills were adequately developed despite these executive errors. Hence, achieving APOS mental structures does not necessarily depend on correct problem-solving responses.

With regard to indefinite integrals, students might follow the procedures and necessary techniques of integration and can solve related problems without knowing why the methods work (Borji & Font, 2019). However, with application problems like kinematics, students must also make sense of the context and problem constraints. Failure to apply the initial conditions in kinematics leads to technical errors. Twenty-one students disregarded direction in the initial velocity and ended up with $c = +7$ for (1). For (3), a similar number of students had challenges with initial displacement, which also was the value of the constant k. Identifying the initial velocity and displacement in the correct perspective is a process skill. Hence, the presence of arbitrary errors signifies inadequate process conception. Kiat (2005) notes that students faltered in finding displacement as they did not consider the change in the direction of the particle. This causes students not to encapsulate the application of integration as efforts to find final velocity and displacement become futile. Finding the final displacement in (3) called for the object-level conception of kinematics, and the presence of the three error types in students' responses shows a lack of coherence among action, process and object conceptions. Therefore the scheme for the application of integration into kinematics was not well-developed. Understanding a concept is constructing a plausible schema, enabling students to assimilate it to what they already know (Iwuanyanwu, 2019). Though learning integrals poses difficulties to students (Langi et al., 2023; Nurhayati et al., 2023), this study managed to delve into the specific aspects of integration that pose conceptual difficulties for students (Czarnocha et al., 2001). Maharaj (2014) posits that students find it difficult to evaluate integrals particularly if they are given in context.

Some instructors have come to terms with students' difficulties in integration and have reacted by teaching integration as a rule or delaying introducing integral calculus as late as possible (Orton, 1983).

This study reveals the stages of students' concept development in the application of integration to kinematics and identifies the associated errors at each stage. There are basically two approaches to solving kinematics problems. These involve instant and average rates of change. Physics students use equations of motion based on average rates of change, while mathematics uses calculus based on instantaneous rates of change. This bifurcation presents a unique challenge to students with a double major in mathematics and physics as they tend to use equations of motion in calculus. However, instantaneous rates of change are given as functions whose turning points require optimization to find unknown quantities. Process and object conceptions of kinematics are called upon to distinguish average and instant rates of change. To physics students, the structural error of using equations of motion in calculus is monumental. Thus, physics students struggle to solve kinematics problems when motion is represented in graphs or functions (Sundari et al., 2023). Average rates of change were observed in non-physics students too, leading to these students computing time as displacement divided by velocity.

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