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Solving algebraic equations by using the bar model: Theoretical and empirical considerations

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Abstract

Solving equations is known to bear several challenges for learners. We discuss an approach based on conceptual understanding regarding the transformation of equations with the help of the so-called bar model in combination with the transposing strategy. First, we sketch shortly the main ideas that guided the development of the learning environment. Second, we discuss insights from the first design experiments with six students working with equation transformation in their regular school curriculum. These design experiments are embedded in a design research approach. In particular, we zoom into the semiotic processes of how learners connect several representations and emphasize a varying difficulty regarding single concept elements necessary to understand the concept of equivalent equations as a whole. Based on that, obstacles that come along with using the bar model are highlighted. Finally, we point to theoretical insights and implications for enhancing our learning environment.

Keywords: algebra, bar model, solving equations, multiple representations, conceptual understanding

INTRODUCTION

Solving algebraic equations is a mathematical procedure that comes with various difficulties for the learners (see for an overview Bush & Karp, 2013; Kieran, 2006). This variety of challenges reflects that several concepts are involved in this procedure. These include a basic understanding of variables as a fundamental prerequisite, the idea of equivalence, a proficient understanding of the equal sign, and knowledge concerning the transformation of algebraic expressions (Weinberg et al., 2016). This combination of conceptual understanding and knowledge of (syntactic) term manipulation poses challenges for learners in school algebra (e.g., Arcavi et al., 2016) that can persist until tertiary education (Weinberg et al., 2016). Several studies point to students' poor use of mathematical symbols without a solid conceptual understanding (e.g., Knuth et al., 2011; Lee & Wheeler, 1989; Sfard & Lichevski, 1994). Other studies associate students' hindrances with a lack of understanding of the meaning of operations in connection with abstract symbols (e.g., Bush & Karp, 2013; Filloy & Rojano, 1989; Herscovics & Linchevski, 1994).

For the MuM-video project from which this paper stems, we started to construct a learning environment for teaching and learning to solve equations while stressing respective conceptual understanding (cf. Prediger & Roos, 2023). The learning environment aims at developing a basic conceptual understanding of expressions, their manipulation, and equivalence while providing a meaning-based language. To this end, we use an adaptation of the so-called bar model (or "model method") from Singapore math (e.g., Hoven & Garelick, 2007) as a promising tool to approach the idea of equivalence in the context of transforming equations. Our work has to be considered topic-specific design research (e.g., Gravemeijer & Cobb, 2006). Thus, it relies on the iterative interplay between designing teaching-learning arrangements, conducting design experiments, and empirically analyzing the processes. In line with this research paradigm, we pursue several goals in this paper: First, we want to show the basic ideas of our usage of the bar model and describe what benefits the

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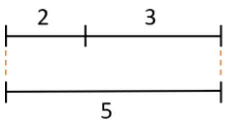
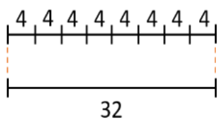
Contribution to the literature

- A subject matter analysis of the bar model combined with the transposing rules for transforming equations is presented.
- Insights into design experiments using the bar model in German school context are given with a focus on semiotic processes.
- In this way, the paper contributes to theorizing on the role of using the bar model in the German school context for developing conceptually based transformations of algebraic equations.

Table 1. Bar model & corresponding equations (following ideas mentioned in Malle, 1993)

Additive bar model	Corresponding equations	Multiplicative bar model	Corresponding equations
	$2 + 3 = 5$ $5 - 2 = 3$ $5 - 3 = 2$		$8 \times 4 = 32$ $32 : 4 = 8$ $32 : 8 = 4$

bar model seems to have for our needs. Second, we want to investigate students' work with the bar model and gain insights for enhancing our learning environment. Finally, we want to summarize the implications for practical and theoretical issues and thus contribute to a theoretical framework for conceptual understanding for using the bar model in the teaching and learning of algebra.

THEORETICAL BACKGROUND

Basic Considerations & Different Approaches for Solving Equations

Solving equations is central in elementary algebra and beyond (e.g., Weinberg et al., 2016). Besides purely procedural perspectives, the transformation steps involved when solving equations can also be supported by conceptual considerations related to the equivalence of equations. Therefore, the balance model is widely used (e.g., Arcavi et al., 2016; Vlassis, 2002). Here, the two parts of each side of the equation sign are considered 'equal' because the analog two parts of a scale are in equilibrium. Equations remain equivalent when a bijective function is used on both sides. Combined with that mode, students learn to follow the 'balance rules' ("do the same thing on both sides"). For example, one might add or subtract the same number/expression on both sides or multiply or divide both sides by the same number/expression (however, with attention to dividing by zero or the variable) because the scale must remain balanced. A conceptual understanding of the basic operations is needed to make the balance model a helpful metaphor for learning to solve algebraic equations. However, Vlassis (2002) showed that this model also comes with a bundle of obstacles. For example, students focus on parts of expressions instead of the whole expressions of the two sides, and the minus sign is not seen in combination with the following expression. In particular, students overgeneralize subtraction as the operation to find the unknown (Filloy & Rojano, 1989). An alternative approach to the balance model, in combination with the balance rules, is the approach of basic transformations. Here, the idea of 'putting an element on the other side of the equal sign by doing the opposite' plays a central role. Again, this procedure leads to equivalent equations (e.g., if A plus B equals C, one might subtract A from C to have B: $A + B = C \Leftrightarrow C - A = B$; analog for multiplication: $A \cdot B = C \Leftrightarrow C : A = B$). Malle (1993) and Selter et al. (2012) favor this approach when introducing equations due to its closeness to learners' intuitive arithmetic thinking. In addition, empirical studies have shown that many learners think of the basic transformations even though they have learned the balance model approach (Kieran, 1988).

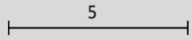
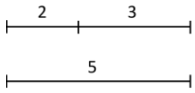
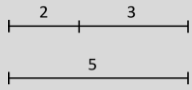
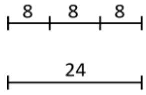
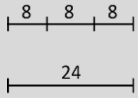
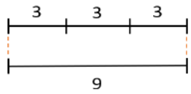
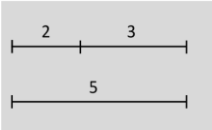
Since there are critical voices concerning the use of the balance model when learning to solve equations (e.g., Booth & Barbieri, 2014; Bush & Karp, 2013; Filloy & Rojano, 1989; Vlassis, 2002), we looked for alternative approaches and possible models for introducing equations with a feature of highlighting the idea of basic transformations.

Bar Model & Corresponding Concept Elements

Malle (1993) proposes the bar model for introducing the transformation of equations because learners can adopt previous experiences with basic operations and their corresponding conceptual meanings. In addition, this model comes with the idea of equivalent equations quite intuitively: Three equations belong together (are equivalent) if they describe the same bar model (Prediger & Roos, 2023). Accordingly, three equivalent equations can be detected (seen) in one bar model and thus be taken from it (see **Table 1**).

The Singapore bar model (e.g., Hoven & Garelick, 2007; Ng & Lee, 2009) works in a similar way. This model is used widely in Singapore's primary schools (e.g., Kaur, 2019) and is discussed to support students' problem-solving activities when working with

Table 2. Concept elements for solving equations and corresponding concept elements in the bar model

Concept elements	Occurrence of concept elements in the bar model	Visualization of concept elements in the bar model
Number	Length of a stripe	
Addition (e.g., 2+3)	Putting together	
Subtraction (e.g., 5-3)	Taking away or determining the difference	
Multiplication (e.g., 3x8)	Counting in units	
Division (e.g., 24:3)	Partitive or quotative model	
Equality (e.g., 3x3=9)	Stripes having the same length	
Inverse operations (e.g., 2+3=5 & 5-3=2)	Doing the opposite	Visible in one bar model, compare examples above
Equivalence	Equations 'belonging together' as describing the same bar model	 2 + 3 = 5 5 − 3 = 2 5 − 2 = 3

arithmetic and algebraic word problem-solving tasks (Baysal & Sevinc, 2022; Ng & Lee, 2005). Research gives first insights into its effectiveness that is all positive (for a review, see Kaur, 2019): the bar model as an external representation that supports the mathematical structure could affect students' performance in mathematical problem-solving (Vicente et al., 2022). Students with difficulties in mathematics have benefitted from drawing the model (Morin et al., 2017). Teachers see it as a helpful tool (Ng & Lee, 2009) that could also serve as a link between arithmetic and algebra (Fan & Zhu, 2007). The bar model could thus support learners in approaching algebra through generalized arithmetic. By using this approach, mental models regarding the operations of addition, subtraction, multiplication, and division established in earlier years can be transferred into algebraic thinking.

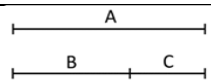
Furthermore, this model used as a line diagram was also shown to be effective for supporting the part-whole schema in translation processes between verbal problems and the symbolic representation (Wolters, 1983).

However, as with every representation, the bar model also comprises several aspects learners have to grasp to be able to work with it properly and understand the underlying mathematical concept (here: equivalent equations). To structure these aspects, we use the idea of concept elements as proposed by Drollinger-Vetter (2011, p. 34). This author refers to concept elements of a mathematical concept as those parts that must be understood to grasp the concept as a whole. Thus, several concept elements have to be understood to grasp the concept of equivalent equations in the bar model. We list corresponding concept elements for equivalent equations, their occurrence in the bar model, and their corresponding visualization in the bar model (**Table 2**).

Since three different equations can be seen in one bar model (see **Table 1**), one operation and its inverse operation can be detected in the same bar model. For example, learners might detect the addition "$2 + 3 = 5$" in the respective bar model since corresponding stripes are placed next to each other. Then, they can change this interpretation by seeing "$5 - 3 = 2$" by making use of the concept element of subtraction as "taking away" of a line segment or "determining the difference" from one line segment to the other – the same counts for multiplication and division. This way, the connection between the equations based on the inverse operation is focused, as the inversion can be immediately seen in one model. However, the dynamic transformation of equations cannot be translated into this way of using the bar model.

The importance of a profound understanding of such concept elements is also highlighted by Ng and Lee (2009). These authors worked with primary students and characterized the challenges they found when working with the bar model as being based on erroneous bar models and misunderstandings of single pieces of

Table 3. Different registers & possible treatments in the context of elementary equations

Symbolic-algebraic	Symbolic-numeric	Verbal-situative	Bar model
$A = B + C$	$5 = 3 + 2$	Eva runs 5 km per week. On Mondays, she runs 3 km, and on Wednesdays, she runs 2 km.	
Examples of treatments in the registers			
$A = B + C$ $\Leftrightarrow A - C = B$	$5 = 3 + 2$ $\Leftrightarrow 5 - 2 = 3$	Eva runs 5 km per week. On Mondays, she runs 3 km, and on Wednesdays, she runs 2 km. This week, she only manages to run on Monday. Thus, she runs 2 km less than in her usual week. This week, she only runs 3 km.	No treatment on level of representation; a transformation is replaced by changing the perspective on the same bar model

information rather than calculation mistakes. Moreover, Yan (2002) emphasized that this model is only helpful when constructed precisely enough to gain the important relationships, e.g., respective concept elements.

Connecting Multiple Representations

Representations play a special and unique role in mathematics. According to Duval (2006), they are not just individual interpretations of a mathematical object but are necessary for constructing knowledge and should not be confused by learners with the actual mathematical object. In the following, we use the concept of representations and its representation register (short: register) proposed by Duval (2006). The use of different representations is viewed as promising for enhancing understanding: "Two symbolic sign systems are used to illuminate each other […]" (Leinhardt et al., 1990, p. 3). A first approach for using several representations in mathematics teaching is given by Bruner's concrete-pictorial-abstract approach (Bruner, 1967). While this approach advocates a sequential introduction of mathematical objects first by using physical objects, then using drawings before going to numbers and symbols, the principle of connecting multiple representations goes even further (Bruner, 1967). Here, the regular change between different representations is focused together with an explicit naming of the connections between them.

In general, one representation (e.g., "A + 2B = C") is embedded in a wider register, which comprises rules for transformations (e.g., the algebraic-symbolic register). The register is important because different aspects of the underlying content become apparent or highlighted due to respective representations and possible transformations. In the context of different registers, Duval (2006) describes two different types of transformations: treatments and conversions:

> "Treatments [...] are transformations of representations that happen within the same register: for example, carrying out a calculation while remaining strictly in the same notation system for representing the numbers, solving an equation or system of equations […]" (p. 111).

> "Conversions are transformations of representation that consist of changing a register without changing the objects being denoted: for example, passing from the algebraic notation for an equation to its graphic representation, passing from the natural language statement of a relationship to its notation using letters, etc." (p. 112).

For our purpose, that is, the context of elementary equations, we consider the following registers: symbolic-algebraic register, symbolic-numerical register, verbal-situative register, and the bar model (as a special kind of iconic representation). We give examples of elements of such registers, possible representations, and respective treatments in **Table 3**.

Here, a change from the symbolic-algebraic representation to the respective bar model would be an example of a conversion.

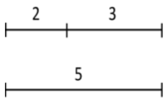
Both transformations (treatments and conversions) are of great importance when learning mathematics, but conversions seem to bear a unique hurdle:

> "Conversion is a representation transformation, which is more complex than treatment because any change of register first requires recognition of the same represented object between two representations whose contents have very often nothing in common. It is like a gap that depends on the starting register and the target register" (Duval, 2006, p. 112).

Several studies have shown that different representations and conscious translations between respective registers support conceptual understanding (e.g., Duval, 2006; Moschkovich, 2013; Schnell & Prediger, 2014). However, the connections between the different representations have to be highlighted and made explicit (Kaput, 1989; Marshall et al., 2010).

As becomes apparent in the quote of Duval (2006) above, a successful conversion is not about the pure

Table 4. Concept elements for connecting a bar model with the symbolic-numeric register

Bar model	Focused concept elements in the bar model	Concept elements as mediator	Symbolic-numeric representation
	• Length of stripes • Putting together • Equal length of stripes	• Length of stripes interpreted as (size of) numbers • Putting together as addition • Resulting length as sum • Equal length as equality • Upper & lower part of bar model as left & right part of equation	2 + 3 = 5

(mechanic) change of representation; it is about connecting the representations.

Therefore, the connections between them have to be explicated. In this sense, we adopt Duval's (2006) idea that a successful conversion "first requires recognition of the same represented object between two representations" (see quote above).

As this recognition is a crucial aspect for many learners, we go into more detail regarding this aspect of "recognition of the same represented object" by using the idea of concept elements. Hence, to perform a conversion (connecting two representations), one has to detect the corresponding concept elements in one representation and use them as mediators for connecting the representations (compare Renkl et al., 2013). We provide an example of such a conversion by connecting a representation from the bar model and the symbolic-numerical register (see **Table 4**).

Although many reasons favor introducing and using several representations, there are also critical voices regarding its use. First, every representation must be considered a learning content, not as self-evident or self-explanatory (e.g., Gravemeijer, 2016).

Second, regarding given representations by the teachers, Fagnant and Vlassis (2013) showed that students tend to guess the corresponding representations in a different register instead of constructing them based on conceptual ideas. From a sociocultural perspective, in particular, the idea of transmitting knowledge through a model given by instruction is criticized (see for an overview Verschaffel et al., 2007). Third, Verschaffel et al. (2007) highlight that all representations have their limits, so they can be considered as serving only to a certain extent.

LEARNING ENVIRONMENT

As alluded to in earlier sections, every representation must be considered a learning content that is neither self-evident nor self-explanatory (e.g., Gravemeijer, 2016). Our learning environment is based on the bar model to promote conceptual understanding and to highlight the idea of equivalent equations in particular. One main characteristic of the environment is that the learners are asked to verbalize the appearance of corresponding concept elements and their mutual relationships between the two registers involved. This way of teaching and learning follows the principle of connecting multiple representations. With this principle as a basis, the learning environment aims to go deeper than the typical procedural character of transforming equations. Conceptual connections between single representations should be established based on the relevant concept elements (**Table 2**). To transform equivalent equations, we orientated our learning environment along the following five steps to systematically guide the work with the bar model.

Step 1 highlights the idea of equality in real-life situations (e.g., weights on a scale or paying money to rent a car). These verbal descriptions are to be matched with corresponding algebraic equations. This first step aims to activate prior knowledge concerning the concept elements of the four basic operations and the concept element of equality in several representation registers.

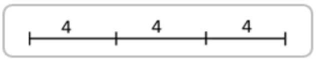
In step 2, the use of stripes (as the basic elements of the bar model) is introduced in the context of running programs. Corresponding stripes represent distances as can be seen **Table 5**.

Here, learners should connect the stripes as iconic representations with the corresponding symbolic and contextual representations. In this sense, first conversions between the bar model and contextual situations and between the bar model and numeric-symbolic representations are required. Accordingly, the following concept elements (compare **Table 2**) are focused on: the size of a number as the length of a stripe, addition as putting together, and multiplication as counting in units. In addition, two running programs can have the same distance in total, which is also true for the corresponding stripes. Hence, the concept element of equality is provided, too.

In step 3, the idea of equivalence of numerical equations is approached by considering three numerical equations that belong to one bar model. Here, the concept element of equivalence is highlighted. The idea of the inverse operation is introduced and must be 'seen' in one bar model to gain the corresponding equivalent equations. Accordingly, the concept elements concerning subtraction and division are provided. Subsequently, the idea of equations describing one bar model (and vice versa) is transferred to algebraic equations in step 4. Step 3 and step 4 aim to identify the corresponding three equations in one bar model. The

Table 5. Some examples of tasks from our learning environment (step 2)

Task: The drawing on the right hand side shows the kilometers covered during each run. Write down the total distance of the run and substantiate this with a suitable calculation.
Calculation:____________________



If I put .... stripes one behind the other and bundle them, it is same as ....

Task:

And what does it look like if I jog a 1 km run and a 2 km run in addition to my three 7 km runs?

Eva:

Monday	7 km
Tuesday	Break
Wednesday	7 km
Thursday	Break
Friday	7 km
Saturday	2 km
Sunday	1 km

- Plot Eva's runs per week on the line below.
- Find a suitable calculation that shows how many kilometers Eva runs in a week.

Calculation:

focus is on the conversions between the bar model and the symbolic representation. All concept elements are made explicit in both representations. The students are led to focus on those equations that belong to the same bar model and thus make sense of the concept of equivalent equations as describing the same situation.

Finally, in step 5, equivalent algebraic equations are focused without considering the bar model. This final step provides the detachment from the bar model. The focus is on the use of the inverse operation. Treatments are considered only in the algebraic-symbolic register.

RESEARCH QUESTIONS

Our development research is framed by a design research approach (Gravemeijer & Cobb, 2006) based on a learning environment that was constructed with a strong focus on the design principle of connecting multiple representations. The learning environment focuses on working with the bar model, which has been shown to have a positive effect on learners' problem-solving skills. In our context, it is used for the concept of equivalent equations. However, critical voices also mind being careful when introducing new representations. Therefore, we want to explore how students cope with the bar model to highlight the idea of equivalent equations. Following our aim of testing and enhancing the learning environment and related theoretical issues, we first focus on challenges that might occur when working with the bar model:

(1) What challenges can be identified in students' work with the bar model in our learning environment?

Furthermore, we are interested in using the bar model to contribute to a conceptual understanding of (equivalent) equations. For this purpose, the bar model and respective transformations (conversions and treatments) play a central role. We thus want to have a deeper look into students' work when transforming equations with the bar model and approach the open question of how far students accomplish the semiotic processes involved in the usage of the bar model:

(2) How can students' semiotic transformations ("conversions" and "treatments") involving the bar model be characterized regarding the usage of the relevant concept elements during students' work in our learning environment?

Finally, the investigation of successful conversions aligns with identifying relevant concept elements in the context of the bar model. Accordingly, we are interested in how far the students can identify corresponding concept elements when working with the bar model:

(3) To what extent are students able to identify the relevant concept elements when working with the bar model in our learning environment?

METHODOLOGY

The research presented in this paper is pursued in a design research methodology because this paradigm combines both the design of innovative instructional approaches according to defined design principles and the investigation of students' work in the designed learning environments (Cobb & Gravemeijer, 2008; Gravemeijer & Prediger, 2019). Our focus lies on gaining deep empirical insights into students' usage of concept elements while connecting multiple representations, thus retracing necessary refinements for the design principle of connecting multiple representations.

Sample & Data Collection

The sample comprises the work of six students from different schools in Germany. All of them have in common that the topic of transforming equations was part of their mathematics curriculum during their respective school year. The sample can be divided into two groups. Group (1), aged 14, worked with transforming equations for the first time. Group (2), aged 18, took part in remediating courses at a vocational school. Group (2) was added to the sample as conceptual

Table 6. Categories summarizing students' challenges when working in our learning environment using the bar model

#	Name of category	Explanation	Illustration (taken from students' work)
Challenges concerning the usage of the bar model			
1	Failure to recognize corresponding mathematical operations in the bar model	Students do not recognize mathematical operations represented in the bar model.	Compare examples discussed below
2	Focus on irrelevant aspects in the bar model	Students focus on irrelevant aspects of the bar model (e.g., what is at top & bottom).	I'm not quite sure myself how I did it, but I copied it from the other drawings and just did the longer line at the bottom. And above just shorter lines.
Challenges resulting from the use of the bar model			
3	Strategy of "solving for variable x" is not developed	In the context of "equations that belong to one bar model" learners do not develop strategy of solving for the variable, they are looking for.	And would not that be easier then to somehow take the equation directly? [Interviewer marks equation $35:5 = f$] ... I do not think so. I find easier, if the- if the f is in it in the, yes how should we say this, within the calculation [points to "$35 = 5 \cdot f$"] and not as a result is there.
4	Overgeneralization "only positive numbers"	Working with the bar model results in only positive values being considered as outcomes. This impacts the assignment of variables & selection & order of operations used.	How do you know that we always did it that way? That we subtracted two from eight? Because otherwise you get a minus number [...]. That one looks at beginning, what largest number is and then subtracts small numbers from the large number … only natural numbers as results.

understanding of the mathematical procedures is particularly crucial for these students. They often have a background of learning procedural elements by heart for upcoming tests instead of developing long-lasting conceptual understanding. Therefore, our learning environment is especially considered helpful for them.

The sessions, given in Zoom, took place from January to July 2021 and were organized in one-to-one sessions with the design experiment leader (the paper's first author). The environment consisted of several tasks and so-called memory boxes, where the learners could record their most important new findings. The participants were asked to think aloud while working on the tasks. The design experiment leader observed students working in the learning environment during the sessions and asked questions to understand their thinking. For example, whenever students' work remained unclear, she asked them to explain their thoughts and gave additional hints when needed. The data comprises a total of 960 minutes of video recordings.

Data Analysis

The video recordings of the design experiments with six students were considered to give insights into students' conversion and treatment processes. We watched all the videos, identified episodes of interest for our investigation (conversion and treatment processes), and transcribed corresponding parts.

For research question (1), we set up a qualitative content analysis (Mayring, 2015) to summarize and structure all the students' challenges observed when working with the bar model in the learning environment. We thus came up with an inductive constructed set of categories (see **Table 6**).

To describe the semiotic processes (research question (2)), we used the transcripts to follow up on how students succeeded in completing the transformations. In the *first step*, we categorized the concept elements (see **Table 2**) used by the students within the processes. Here, we interpreted the corresponding linguistic utterances of the learners as occurrences (or omissions) of concept elements. In the *second step*, we looked for indications in the transcripts of how students succeeded in using the concept elements as mediators in the different representations. In the *third step*, we constructed a table to make the underlying processes visible. Therefore, we put the corresponding excerpts from the transcript in the first column. In the second column, we list the focused concept elements belonging to the first representation apparent in this excerpt. The third column is about the possible use of these elements as mediators between representations one and two. This second representation is given in the last column of the table.

If we could not detect any explicated usage of the concept elements as mediators, we tried to describe students' strategies for performing the change of representation. We summarized the different strategies in the course of a qualitative content analysis and thus came up with an inductive set of categories.

We analyzed students' explanations of their work to investigate the treatments involved. We cite episodes from the transcripts, summarize the approach observed, and state our interpretation of the episode.

To answer research question (3), we looked at the parts of the videos and transcripts, where students were working with the bar model. Based on the students'

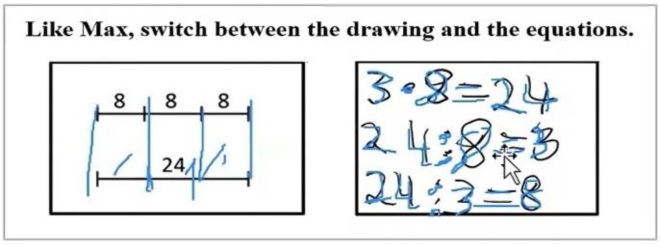


Figure 1. Anno's task & solution (Source: Screenshot from the video session)

utterances, we coded evolving concept elements as identified respectively focused by the student based on the descriptions in **Table 2**. For example, Vivien explains how she arrived at the equation "4 × 8 = 32" using the bar model: "that's ... eight fours, and then just eight times four is 32". Here, we have coded the concept element of multiplication (as counting in units) as identified by Vivien. Critical cases were discussed between the authors until intercoder agreement was achieved in the sense of consensual validation (Flick, 2007).

RESULTS

This section will first list the results concerning the overarching challenges we found when students worked in the learning environment. Subsequently, individual students are discussed with emblematic examples of conversions and treatments. To this end, relevant excerpts from the transcripts are given and analyzed, focusing on respective transformations. Finally, the results presented in this section will be used to answer our research questions.

Results Concerning Arising Challenges When Working With the Bar Model

The qualitative content analysis led to the categorization of the four challenges listed in **Table 6**. By following our inductive approach, we came up with two meta-categories for structuring the four subcategories: challenges concerning the usage of the bar model and challenges resulting from the use of the bar model.

In category 1, we summarized students' difficulties recognizing corresponding mathematical operations in the bar model. We will discuss several examples of this phenomenon below. Category 2 comprises students' challenges when focusing on irrelevant aspects in the bar model and thus failing to recognize essential (concept) elements for working with the given task. Such irrelevant aspects include the bars' order (what is written on the upper bar and the lower bar), and the concrete bars' length when using variables. When being asked to work with the bar model, such challenges prevent a viable approach to work with the bar model.

Moreover, we found challenges resulting from the use of the bar. Working with the bar model in the present form of the learning environment did not lead to the strategy of "solving for the variable x" (category 3). Finally, exposure to the bar model led learners to implicitly believe that only positive values occur (category 4).

Thus, challenges concerning the usage of the bar model directly concern the work with the model, whereas challenges resulting from the use of the bar model concern rather the process of solving equations per se.

Results Concerning Conversions

This subsection presents the analysis of three cases of conversions that we found paradigmatic in our sample.

Case Anno: Intended conversion: Bar model–Symbolic-numerical representation

Anno is 14 years old, attending the 8th grade of a comprehensive school, and can be classified as an average student based on his grades. The episode shown below is part of the second session with Anno and belongs to step 3 of the learning environment. Anno is asked to set up the corresponding three equations on the symbolic level, starting from the given bar model. The task and Anno's solution are shown in **Figure 1**.

After Anno has written the three equations, the following episode takes place:

> I: Can you explain to me, where you see the first equation in the drawing, this three times eight equals 24?
>
> A: Yes, this ehm so 24 is this long line here [...] And these here are three 8s lines, like three wooden parts that one has put together or have the same length as the 24.
>
> I : Yes. Great. And this second equation, that with the division, do you see that somewhere in the drawing as well?
>
> A: If I have a lot of such small 8s things here and would put them into the 24, just like one, so 1, 2, and 3, I would just eh calculate, that I would also–come to eight […] that you see here, if you lengthen the lines here, you also get three times, therefore three parts.

Our analysis of the episode led to the results summarized in **Table 7**.

In this episode, Anno focuses on different concept elements in the bar model, listed in the second column of **Table 7**. His reports on these elements fit the equations he mentions in the symbolic-numeric representation. In our interpretation, Anno uses these concept elements as mediators to link both representations meaningfully, even though he does not

Table 7. Analysis of conversion performed by Anno

Anno's description of the bar model	Focused concept elements in bar model	Mediating concept elements	Symbolic-numerical representation
Yes, this ehm so 24 is this long line here [...]. And these here are three 8s lines, like three wooden parts that one has put together or in end have same length as 24.	• Length of stripes • Putting together • Equal length of stripes	• Length of stripes interpreted as size of numbers • Putting together as addition • Repeated addition & counting in units as multiplication • Resulting length as sum • Equal length as equality	$3 \times 8 = 24$
If I have a lot of such small 8s things here & would put them into 24, just like 1, so 1, 2, 3, I would just eh calculate, that I would also–come to eight.	• Put smaller stripes into a longer one • Count number of smaller stripes used	• Put smaller stripes into a longer one as "fitting in"/quotative model of division	$24 : 8 = 3$

Figure 2. Frank's bar model & corresponding equations (Source: Excerpt from the learning environment)

mention the links explicitly in every case (not explicitly mentioned: written in grey in **Table 7**).

According to our interpretation, Anno succeeds in connecting representations via the following concepts elements: connecting the length of the stripes with the size of the numbers, interpreting "putting together" as addition, and connected with that, "repeated addition and counting in units" as multiplication, "putting smaller stripes into a longer one," and "counting these units" as division.

Anno's conversion process from the bar model to the symbolic-numerical representation is based on his identification of concept elements in the bar model. He is able to use these concept elements as mediators to construct the corresponding symbolic-numerical representation and thus accomplishes the conversion.

Case Frank (1): Intended conversion: Bar model–Symbolic-numerical representation

Frank is an 18-year-old student at a prevocational school. The episode below is part of the second session with Frank and stems from the beginning of step 2 of the learning environment. Here, Frank is asked about the connection between the symbolic representation of the equations and the corresponding bar model (see **Figure 2**). While the equations on the right side can be interpreted as results of treatments, the learners should justify them by a conversion involving the bar model. All three equations can be seen in the given bar model.

In this context, the following dialogue takes place between Frank and the interviewer:

F: Yes, two and three equals five [...] so that was the drawing, and then as a calculation, yes, two plus three equals five. Then you always had to make the number, uh, a task, where the number[s] are the result again. For example, just again five minus three is equal to two, the two has also been displayed. Five minus two is, yes, the three, the three is also there and with two plus three is, yes, also the five is shown there, therefore.

I: And did the picture help there in any way, or what did the picture do in the process?

I understand the calculation. But what does the picture do in the process?

F: Uh, the picture shows then, which numbers you can or must use, in this sense […] You have to use the two, the three and the five to write the ... when you write the calculation.

I: And do you see this operator there as well, so does this plus or minus, can you see that there somehow in the, in the drawing as well?

F: No, you cannot ... so I would say, so ... I do not know what others would see, so I would not see any plus or minus there now. Because this, this line between, at the two and the three can yes–uh is simply the, the ratio of the two and the three or how you would represent it.

We analyze this episode, as follows (see **Table 8**):

Frank does not identify meaningful (concept) elements in the bar model. When asked about the operations displayed, he could not name the respective elements in the bar model. (Here, the interpretation is possible that Frank tries to interpret the short vertical line between the "2" and "3" in the bar model in combination with the horizontal line as some kind of "+"-sign (compare episode below), but he does not follow this interpretation.) Frank does not identify the mathematical operations in the bar model and thus does not mention any corresponding concept element like "putting together" or "placing pieces one behind the

Table 8. Analysis of conversion performed by Frank

Frank's description of the bar model	Focused (concept) elements in bar model	Mediating concept elements	Symbolic-numerical representation
Yes, 2 and 3 equals 5 [...] so that was the drawing, and then as a calculation, yes, 2 plus 3 equals 5.	• Numbers written on the stripes	"Strategy": Take numbers from the picture and make tasks with them. Every number should serve as a result once.	$2 + 3 = 5$
Then you always had to make the number, uh, a task, where the number[s] are the result again. For example, just again 5 minus 3 is equal to 2, the 2 has also been displayed. [...] Uh, the picture shows then, which numbers you can or must use, in this sense […]. Because this, this line between, at 2 and 3 can yes- uh is simply the, the ratio of 2 and 3 or how you would represent it.	• Numbers written on the stripes • Ratio of the two segments on the upper stripe		$5 - 3 = 2$ $5 - 2 = 3$

Table 9. Analysis of conversion performed by Frank (episode 2)

Frank's description	Focused (concept) elements	Mediating concept elements	Bar model
Eva runs 3 times, 3 is there, and 7 km is also another 7, it's also there. So that's just 3 times 7 […]. So, I see there no "times" directly, that could also be a plus, that could also be just, if I do not see the text, it could also be plus.	• Numbers given in situation • Focus on iconic similarities to "plus" or "times" sign (?)	"Strategy": Looking for the same numbers in both representations to find the match.	

other". In addition to the pure numbers displayed, he also mentions the aspect of ratio, which can lead to difficulties in the future, especially when dealing with variables. Since he does not identify significant concept elements in the bar model, he uses an evasion strategy for performing the conversion: He only focuses on the numbers displayed in the bar model to pass to the symbolic-numerical representation. Frank changes their order to gain different equations and uses corresponding operations to get the result.

Case Frank (2): Intended conversion: Verbal-situative–Bar model

We cite another episode with Frank to highlight his difficulties in recognizing concept elements concerning mathematical operations in the bar model (Table 9). Frank has to choose the bar model fitting the following description of "Eva's run" in this task:

> "Eva runs seven km three times a week. Max runs three km every day from his parents' to his grandparent's house. Andi runs 13 km on Tuesdays and seven km on Thursdays."

The corresponding bar model is shown in Figure 3 (bottom of Figure 3). However, Frank chooses the wrong bar model (upper solution in Figure 3) and explains his choice as follows:

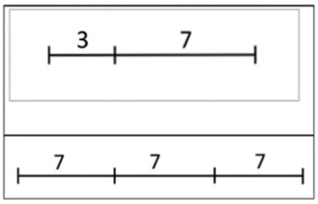


Figure 3. Two different bar models that might fit "Eva's run" (Source: Excerpt from the learning environment)

> F: Eva runs three times, the three is there, and seven km is also another seven, it's also there. So that's just three times seven.
>
> I: And could you explain, where you see the "times" in the drawing?
>
> F: I just say it. With me it's like that, I just say it, three times seven, this is a habit for me, that I say it like that. So, I see there no "times" directly, that could also be a plus, that could also be just, if I do not see the text, it could also be plus […].
>
> F: Because it looks like plus simply because it's not ... with the others uh several times the numbers or it is just completely different there. It just looks like this, because three and seven also results in, could also result in 10. And this [referring to the context], if I did not have this in my mind, this three times a week seven km run,

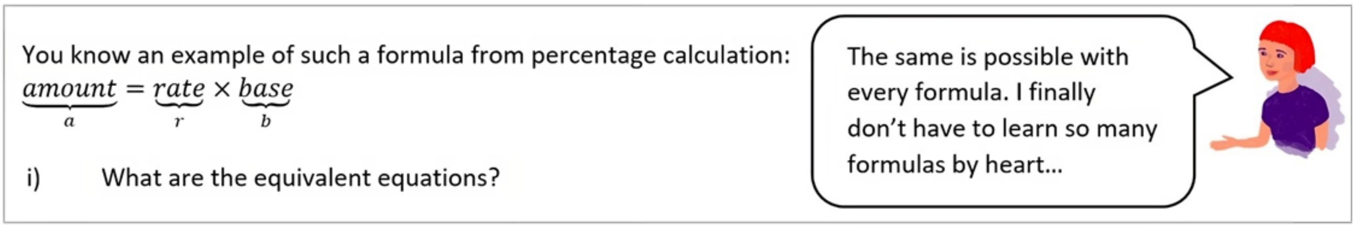


Figure 4. "Tessi's" task (Source: Excerpt from the learning environment)

> then I would know, then I would not know it either, like this. This text is simply this three times, so three times seven km running in the week.

Again, Frank focuses on the numbers used in the text and the numbers in the bar model. When asked about the "times" in the drawing, it is unclear in his response if he is talking about the meaning of times as an operation or about the sign ("×"). Frank does not identify any concept element in the bar model in this episode. Here, another evasive strategy for performing conversion becomes apparent: looking for the same numbers in both representations.

Results Concerning Treatments

Case Tessi: Treatment in the algebraic-symbolic register

Tessi is 14 years old and has good grades at secondary school (8th grade). The following episode is part of the second session with her and is located at step 5 of the learning environment. A learner can only solve the task shown in **Figure 4**, after a detachment from the bar model has taken place.

After Tessi states corresponding equations ($\frac{a}{r} = b, \frac{a}{b} = r$), she explains her finding of the solution as follows:

> (lines 1-6) T: Amount divided by rate is the base and amount divided by base is the rate. But I really have no idea, as I said, but that's how I could imagine it, according to the scheme we've done so far [referring to the tasks working with the bar model].

> (lines 7-9) I: Yes, that's definitely correct [referring to the three equations]. But can you explain to me how you came up with it?

> (lines 10-15) T: We have the full length of this amount and then so and so often the percentage, what is given by the number of the base. So for example 10 times as much, so 10 times this percentage and this 10 would then be the base, so, or, yes.

> (lines 16-22) […] We still have these three components, which we need, and that's why we have to divide this one, this really high amount, which is probably the highest number, by one of the smaller ones to get another one and then the other way around, so divide by the one to get the other one.

In this case, we do not include a table for analysis since we do not want to examine a conversion but rather describe the treatment undertaken.

Even though this episode occurs after the (desired) detachment process from the bar model, it demonstrates a strong connection to previous work with the model. When explaining her way of finding the equations, Tessi explicitly refers to the bar model: She identifies the amount with the "full length" which is equal to "base times rate" (line 10 - 15). For gaining the other two equations, she uses the strategy of dividing the highest number by a smaller one (line 16 - 21). Thus, she gets two more equations. In this episode, ideas resulting from the bar model help Tessi to find the equivalent equations. However, her strategy bears some hindrances. For example, the idea of having the highest number that is equal to the product of two smaller numbers is limited to cases with factors bigger than 1.

Results Concerning Concept Elements Identified by Students

We give an overview of all concept elements identified by the students in our study in **Table 10**.

All students identified at least some concept elements concerning basic operations in the context of the bar model. However, grasping (the concept element of) division in the bar model (partitive or quotative model) seems delicate. The gaps in **Table 10** indicate that no student elaborated on every concept when explaining his/her solution. Furthermore, we could not code a concept element concerning "inverse operation" as identified since students seemed to rely heavily on the numerical or algebraic equations when explaining the phenomenon of 'belonging together'. At no point did the students explain this phenomenon with respect to the bar model. As these concept elements are missing in every case, we could not identify the overarching concept of equivalence.

SUMMARY & DISCUSSION

Answering research question (1) ("What challenges can be identified in students' work with the bar model in our learning environment?"):

Table 10. Summary of concept elements identified by learners ("CE": Concept element & "c": Concept)

#		Costas	Tim	Vivien*	Frank	Anno	Tessi
Basic interpretations							
CE 1	Size of numbers as length of lines	X	X		X	X	
CE 2	Equality as having same length	X	X	X	X	X	X
Basic operations							
CE 3	Addition as putting together	X	X	(X)	X		X
CE 4	Subtraction as taking away or determining the difference	X		(X)		X	
CE 5	Multiplication as counting in units or repeated addition	X	X	X	X	X	X
CE 6	Division as partitive or quotative model				X	X	X
Idea of inverse operation							
CE 7a	Inverse operation–Doing the opposite (addition vs. subtraction)						
CE 7b	Inverse operation–Doing the opposite (multiplication vs. division)						
Concept of equivalent equations							
c	Equivalence–Belonging to one bar model						

Note. *In one episode, Vivien gives a somewhat incoherent explanation of her solution ("line below ... 7 cm maybe … and above 4 cm cut off and 3 cm sticked or something like this"); even though Vivien does not show any identification of addition and subtraction in the context of the bar model in other episodes, this episode seems to highlight some basic ideas about corresponding concept elements, and accordingly, we marked corresponding cells with "(X)".

Using a qualitative content analysis, we came up with two meta-categories of challenges:

(i) challenges concerning the usage of the bar model and

(ii) challenges resulting from the use of the bar model.

For the *first meta-category*, we noticed students' failure to recognize corresponding mathematical operations in the bar model and their focus on irrelevant aspects in the bar model. These aspects were surprising for us and are discussed in more detail in the context of research question 2. The difficulties resulting from the use of the bar model are categorized in the *second meta-category*. Here, we categorized the phenomenon that the strategy of solving for the variable x is not developed, and we found some overgeneralizations summarized with the heading "only positive numbers". For example, using stripes (with positive length) fostered overgeneralizations about what numbers the variables stand for and the selection and order of operations used. As every representation has its limits and can only serve to a certain extent (e.g., Verschaffel et al., 2007), the difficulties we found can be seen as characteristic of the work with the bar model similar to the hindrances Vlassis (2002) highlighted for the case of the balance model.

Answering research question (2) ("How can students' transformations ("conversions" and "treatments") involving the bar model be characterized during students' work in our learning environment?"):

Concerning the transformation of conversion, we found several students performing a successful conversion in the sense of Duval (2006). We showed an example, where the student Anno succeeded in focusing on the necessary concept elements in the bar model, thus connecting this representation with the corresponding one in the symbolic-numerical register. However, we also saw students who did not complete such conversions by connecting the representations involved but using evasive strategies like "just take the numbers from the picture and construct tasks with them" or "look for the same numbers in both representations to find the match". Again, the mere occurrence of evasive strategies is not surprising. Related phenomena are already known in other areas of elementary algebra (e.g., Prediger, 2008). Here, the question arises: what prevents a successful conversion? As explained above, we consider a successful conversion as the connection of corresponding representations. For this to happen, the concept elements in the representations must be captured and used as mediators between the representations involved.

In this case, the bar model can be seen from two perspectives: On the one hand, it is a learning objective in itself, as the meaning of its single elements has to be understood to see them in the light of the necessary concept elements. On the other hand, the bar model serves as a tool for uncovering existing gaps regarding essential concept elements (see also Ng & Lee, 2005). In particular, concept elements related to mathematical operations (addition, subtraction, multiplication, and division) may not have been sufficiently internalized. Thus, significant gaps in students' pre-knowledge might exist, hindering them from learning new mathematical concepts. Therefore, using the bar model could give important hints regarding essential foundations of understanding that require consolidation before moving forward with additional content.

As an example of treatment in the algebraic-symbolic register, we discussed the work of Tessi. This student performs the treatment by referring to ideas taken from the bar model. She identifies the product of the multiplication with the length of the longest stripe and the two factors with the smaller stripes. She uses the strategy of dividing the highest number by a smaller one

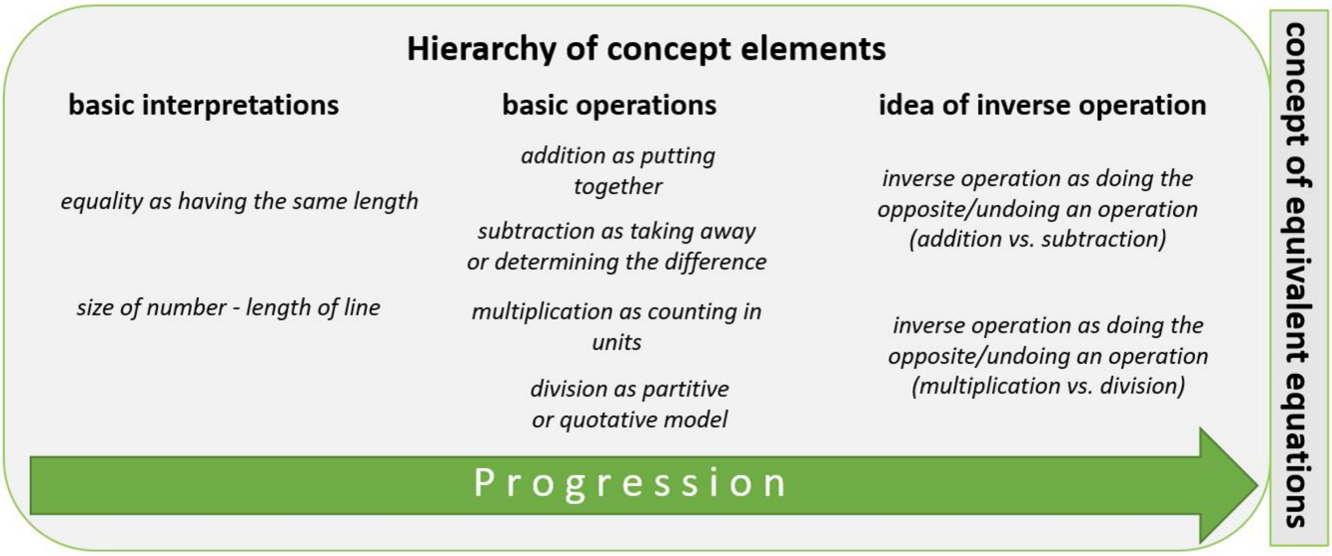


Figure 5. Hierarchy of concept elements (Source: Authors' own elaboration)

to gain the other two equations. As known from previous research, the transition from natural numbers to rational numbers comes with distinct obstacles, e.g., the idea that 'multiplication makes bigger' cannot be transferred from natural numbers to rational numbers (e.g., Prediger, 2008). As with the balance model, the bar model also seems to bear its obstacles and challenges that must be considered when working with it.

Answering research question (3) ("To what extent are students able to identify the relevant concept elements when working with the bar model in our learning environment?"):

As highlighted in the sections above, students identified corresponding concept elements in the bar model differently. Nearly all students seemed to grasp the concept elements of basic interpretations in the bar model (interpretation of length of lines as numbers and equal length of lines as equality). However, the interpretation of division in the bar model as partitive or quotative model is more delicate. Moreover, we could not detect concept elements concerning the inverse operation in the bar model, i.e., the students failed to grasp the concept of equivalent equations as a whole. They focused rather on the single steps of transforming (like adding, subtracting, …) than their combination. The phenomenon of not identifying concept elements can be due to not existing pre-knowledge concerning those concept elements or to the unknown representation of the bar model itself. However, how learners identified concept elements seems to correspond to a progression concerning concept elements in the bar model (see **Figure 5**).

In discussing our research, we must highlight the characteristics and limitations: First, our experiments took place during Zoom sessions. Although the students were more or less used to them due to the pandemic, it brought some challenges: Although we believe that implementing the bar model in a virtual environment did not influence the emergence of the categories of challenges summarized in **Table 6**, a face-to-face conversation about them might have been beneficial. For example, gestures regarding the differentiation of the single operations could be easier to grasp in a face-to-face situation. Second, in considering the different ways students made conversions, we neither distinguished between the directions in which conversions occurred nor did we focus on the epistemic role of language as done, for example, in Uribe and Prediger (2021). Post and Prediger (2022) characterize four semiotic processes in dealing with multiple representations according to their degree of integration of representations. Following these authors, "explaining connections between representations explicitly by naming the correspondance of elements (and enventually justifying the correspondence)" (Post & Prediger, 2022, p. 101) is considered to be the highest degree of integration of representations. Considering that our analysis was based on students' explanations of such connections, it becomes clear that our approach posed high demands on the learners. Third, our analysis is based on the verbal articulations of the learners. Nevertheless, we are aware that the actual thinking of the learners could differ.

Based on these results, we can discuss the supposed benefits of the bar model. The bar model is highlighted as a valuable tool for teaching and learning mathematics at different ages (e.g., Kaur, 2019; Koleza, 2015; Ng & Lee, 2009). However, we detected several challenges when working with the bar model in the context of transforming equations with students in grade 8 or even higher in prevocational settings. Hence, the bar model for the learning of transforming equations is, on the one hand, a learning object per se. As with the balance model, the bar model also comes with specific obstacles and possible overgeneralizations within its detachment process. On the other hand, it is a promising tool to support conceptual understanding. It provides a way to make sense of transforming equations and the concept of equivalent equations. In addition, such challenges in dealing with a representation respectively several representations are not surprising (Goldin & Shteingold, 2001; Knuth, 2000). It becomes clear that such (new) representations need to be carefully introduced while negotiating their meaning (Duval, 2006).

These insights point to possible enhancements of our learning environment and theoretical implications. We will summarize these aspects in the following section.

CONCLUSIONS

We summarize implications for a local theory concerning learning about equivalent equations with the bar model and practical implications for our learning environment.

When working with the bar model, a profound understanding of basic mathematical operations has to be considered essential. This understanding is not only necessary for performing respective operations when solving equations in arithmetical or algebraic contexts; learners need to have a proper conceptual understanding to understand which operations are represented in the bar model or perform operations in the bar model themselves. In this sense, the bar model must first be seen as a learning object in its own right before it can aid learning. Accordingly, this model is neither self-evident nor self-explanatory, as learners need specific knowledge to work with it. However, the bar model can also be seen as a tool to make the missing conceptual understanding for transforming equations visible and serve as a starting point for developing it. In our investigation, we have elaborated on the various concept elements that constitute the concept of equivalent equation in the context of the bar model. This nuanced analysis seems helpful in guiding the teaching-learning processes and diagnosis. In this respect, highlighting the challenges we conceptualized also seems significant. Moreover, we have provided the first approaches for describing successful conversion in algebra when working with the bar model. In addition, some evasive strategies became visible. Most specifically, we see far-reaching possibilities for further research on this topic.

To sum it up, the bar model has to be introduced carefully. Concerning the practical implications, we constructed an introduction part, where the learners get to know the bar model and respective basic concept elements (basic interpretations, basic operations see **Figure 5**). In that section, we also included tasks, where learners could repeat or learn about the basic meaning of the basic operations involved ("addition as putting together" and so on), as these interpretations represent indisputable pre-knowledge for working with the bar model. Concerning the conversions involved in the learning environment, we systematized the possible combinations and directions between the semiotic registers involved and included respective combinations. In addition, we included scaffolds (e.g., language patterns) to provide additional help with the conversions to be done deliberately. As we have seen above, grasping the idea of division in the bar model (partitive or quotative model) bears some hurdles for our learners. Thus, we included additional tasks to spend more time on this operation and its meaning and appearance in the bar model.

Another obstacle learners faced was that the strategy "solving for the variable x" was not developed in the learning environment. This could be a consequence of the fact that with the basic transformation rules no treatment in the bar model is possible; thus, students could not solve for the variable x in this register. Therefore, we changed our approach and applied the so-called balance rules ("doing the same on both sides") to the bar model. Accordingly, the bar model is no longer a 'static' representation; the same operations can now be performed on both sides (lengthening the lines, shortening, multiplying, ...) as long as the two lines remain the same length. In this way, we can also perform treatments in the bar model and thus directly correspond the transformations of the bar models to the procedure for solving equations in the symbolic register (compare Tondorf & Prediger, 2022 for the case of expressions). The idea of equivalent equations describing the same bar model is thus shifted to the idea that equivalent transformations leave the stripes with the same length (however, with a possible change of this length).

Finally, the detachment process and possible overgeneralizations had to be tackled. Therefore, we included tasks that prepare or accompany the detachment from the bar model. We have also included tasks (e.g., including negative numbers) designed to counteract the overgeneralizations that have been highlighted above. This way, we integrated tasks, where the bar model is a helpful tool and others, where the bar model cannot be applied anymore (see also Ho & Lowrie, 2014). In line with design research, we will start a new cycle of testing and investigating our learning environment after revising it to expand our findings for theory and practice.

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