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Quadrilaterals hierarchical classification and properties of the diagonals: A study with pre-service mathematics teachers

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Abstract
The aim of this research is to investigate the main characteristics of the processes of constructing a hierarchical classification of quadrilaterals and identifying properties of their diagonals by pre-service middle and secondary school mathematics teachers (PMTs), when involved in whole-class discussions with the teacher educators, after having solved tasks focused on those topics. Data collection comprised video recordings of the sessions and PMTs' written work, analyzed qualitatively. Findings indicate that constructing the hierarchical classification of quadrilaterals involved the PMTs in prototypical judgments, and dichotomous comparisons. In comparison the identification of the diagonals' properties was influenced more by definitions and logical relationships, reflected in judgments and comparisons. It was concluded that participation in these processes, with whole-class discussions and development of schemes to illustrate inclusion relationships, may assist PMTs with prototypical phenomenon and dichotomous comparisons, benefiting their future teaching practice.

Keywords: pre-service mathematics teachers, hierarchical classification of quadrilaterals, prototypical phenomenon, dichotomous comparison

INTRODUCTION

Understanding how teachers learn and comprehend specific mathematical concepts, such as the hierarchical classification of quadrilaterals, can promote the development of approaches that enable them to identify and address obstacles related to the teaching of this topic in geometry. In a hierarchical classification, the more specific concepts are viewed as subsets of more general concepts, which allows for simplifying the systematization of concepts and deductive processes (de Villiers, 1994). This differs from partitive classification, for example, which considers subsets of concepts as disjoint. Furthermore, classification involves identifying properties and defining concepts, creating a relationship of dependency between the processes of defining and classifying (de Villiers, 1994).

Regarding quadrilaterals, the processes of classifying and defining are complex and pose challenges for both students (Haj Yahya et al., 2024) and pre-service teachers (Avcu, 2023; Brunheira & Ponte, 2019; Fujita, 2012; Fujita & Jones, 2007; Miller, 2018; Ulger & Broutin, 2017; Zazkin & Leikin, 2008). Research shows that having difficulties with the content can lead pre-service teachers to feel unprepared to teach geometry (Niyukuri et al., 2020).

One of the main difficulties with the hierarchical classification in geometry is related to the prevalence of examples considered prototypes of concepts (Fujita, 2012). One possibility to overcome that difficulty is to promote dialectical learning, based on interactions and opportunities to construct concepts (Naftaliev & Hershkowitz, 2021). However, the influence of prototypical examples in the hierarchical classification of quadrilaterals is still observed even in an exploratory and interactive context with pre-service teachers (Brunheira & Ponte, 2019). According to Haj Yahya et al. (2024), an effective alternative for identifying hierarchical relations between parallelograms, rhombuses, rectangles, and squares is to construct schemes in the form of conceptual maps.

In this sense, it is still necessary to investigate approaches for the hierarchical classification of

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Contribution to the literature

- This article contributes theoretically by identifying, beyond the interference of prototypical phenomenon in the hierarchical classification of quadrilaterals, the presence of a phenomenon we term "dichotomous comparison" between quadrilaterals.
- The study reinforces the importance of involving pre-service teachers in the process of defining and classifying in the scope of hierarchical classification in geometry, even when apparently simple concepts are involved.
- It shows the role of whole-class discussions in teacher education.

quadrilaterals with pre-service mathematics teachers (PMTs) that may reveal alternatives for overcoming difficulties, such as those related to the prototypical phenomenon. Interactive approaches coupled with the development of schemes for the hierarchical classification of quadrilaterals, including trapezoids and kites, in addition to parallelograms, and allowing for the exploration of various properties, such as those of the diagonals of quadrilaterals, can unveil potential features for geometry teaching. Thus, in this study, we investigate characteristics of the processes of constructing a hierarchical classification of quadrilaterals and identifying properties of their diagonals with pre-service middle and secondary school mathematics teachers. We aim to address the following research questions:

(i) What are the characteristics of the process for constructing the hierarchical classification of quadrilaterals, involving whole-class discussions and the development of a hierarchical relations scheme with PMTs? and

(ii) What are the characteristics of the process for identifying properties of quadrilateral diagonals, involving whole-class discussions, and based on a hierarchical relations scheme, with PMTs?

THEORETICAL FRAMEWORK

Classifying and Defining in Geometry

Classification involves identifying similarities among objects, considering a specific attribute, even if they have differences, in order to establish equivalence between these objects (Mariotti & Fischbein, 1997). Classification can also be understood as a process of mathematical reasoning "that infers, by the search for similarities and differences between mathematical objects, a narrative about a class of objects based on mathematical properties and definitions" (Jeannotte & Kieran, 2017, p. 11). Thus, one of the processes of mathematical reasoning associated with classification is comparison.

In general, classifying involves identifying properties of objects, and in the context of mathematics, definitions express the properties that characterize the objects (Mariotti & Fischbein, 1997). Thus, classifying objects requires defining them in some way, just as defining involves classifying (de Villiers, 1994). According to Alcock and Simpson (2017), classification is better developed when definition tasks are proposed prior to it, and it is further enhanced when the definition task is based on prompts that ask for an explanation of a certain concept, rather than asking for a definition or presenting definitions.

A definition is correct if it contains necessary properties (applicable to all elements of the set) and sufficient properties (whenever they are satisfied, all elements of the set can be obtained), and it is economical insofar as it contains a minimum of necessary and sufficient properties without superfluous or redundant information (de Villiers et al., 2009). However, such characteristics of the defining process may involve some obstacles. For example, Zazkin and Leikin (2008) report difficulties among pre-service secondary teachers with definitions of a square. According to the authors, this does not mean that the pre-service teachers do not know what a square is, but rather that some of them struggle to distinguish necessary and sufficient conditions or to use appropriate mathematical terminology. In Avcu (2023), in addition to pre-service middle school mathematics teachers having difficulties in defining a square and rectangle economically, the linguistic structure of the terms "trapezium" or "kite" did not allow these PMTs to establish a relationship between their native language and the language of geometry, making it difficult to identify properties of these quadrilaterals.

A structured approach to mathematics may convey the existence of a single correct definition for each mathematical object (de Villiers et al., 2009). However, defining is a process that can vary depending on the situation (Mariotti & Fischbein, 1997), and one possibility of variation is to consider hierarchical (inclusive) definitions, which allow "the inclusion of more particular concepts as subsets of the more general concept" (de Villiers et al., 2009, p. 191), or partitive (exclusive) definitions, in which "the concepts involved are considered disjoint from each other (i.e., squares are not considered rectangles)" (de Villiers et al., 2009, p. 191). These two possibilities of definition imply, respectively, hierarchical (inclusive) and partitive (exclusive) classifications. Since both are correct, the choice between one or the other may be a matter of convenience and a curriculum designers' choice. However, hierarchical classifications are more common because they are more functional, enabling the use of economical definitions, simplify the systematization of concepts and deductive processes, provide conceptual frameworks, and allow for a global perspective of the situation (de Villiers, 1994), although they may be more challenging because they require logical deduction (Fujita & Jones, 2007). For example, in Miller's (2018) study, most of the quadrilaterals' definitions presented by pre-service elementary school teachers contained necessary but not sufficient or minimal characteristics, which made it difficult to construct hierarchical definitions.

To discuss issues related to defining and classifying geometric concepts, we find it pertinent to address aspects concerning the understanding of concepts. According to Vinner (1991), the understanding of a concept is permeated by the construction of a concept image. Therefore, in the next section, we will address the process of constructing a concept image and associated aspects.

Concept Image and Concept Definition

A concept image is a cognitive structure that includes the mental image of a particular concept, its properties (Vinner & Hershkowitz, 1980), impressions, and experiences, that is, what is evoked by memory when we think about the concept (Vinner, 1991). The concept image is not static, rather it is built and transformed throughout the experiences that the individual goes through (Tall & Vinner, 1981).

A concept definition, on the other hand, is a verbal definition that explains the concept (Vinner & Hershkowitz, 1980). According to Tall and Vinner (1981, p. 2), the concept definition "may be learnt by an individual in a rote fashion or more meaningfully learnt and related to a greater or lesser degree to the concept as a whole", and it can be a personal reconstruction of the definition by the student themselves, that is, an explanation of their own concept image. Thus, the concept definition may vary; in other words, the personal concept definition may differ from what is formally accepted by the mathematical community (Tall & Vinner, 1981).

Even though the definition can assist in the construction of the concept image, it acts as scaffolding, because once it is formed, the definition becomes dispensable, remaining inactive or forgotten, as it is usually the concept image that is referred to (Vinner, 1991). In reality, there are three possible situations when a student encounters a definition of a concept that does not match his/her concept image for that concept:

(i) his/her concept image may be altered to include the new information;

(ii) the concept image is not altered, and the definition will be forgotten or distorted over time; and

(iii) the concept image is not altered, but the definition will be repeated by the student when asked about the concept, even though in all other situations the concept image is his/her reference (Vinner, 1991).

In this sense, particularly in the third situation, the same individual may react differently to the same concept. As a matter of fact, the concept image does not need to be coherent in all situations as according to Tall and Vinner (1981) "different stimuli can activate different parts of the concept image" (p. 2). The part of the concept image that is activated in a given situation is called the evoked concept image, and if different parts are activated simultaneously, cognitive conflicts may arise. Furthermore, in certain situations, cognitive conflict factors may be evoked subconsciously, generating the sensation that something is wrong (Tall & Vinner, 1981).

Although geometric concepts are defined based on their attributes, this is not sufficient to describe how the cognitive development of the concept image occurs in the mind. One of the characteristics of this development is the prototypical phenomenon (Hershkowitz, 1990), as explained in the next section.

The Prototypical Phenomenon in Geometry

Concepts possess "relevant (critical) attributes (those attributes that an instance must have in order to be a concept example) and noncritical attributes (those attributes that only some of the concept examples possess)" (Hershkowitz, 1990, p. 81), along with certain examples that represent them, which are mathematically equivalent in that they satisfy the definition of the concept but differ visually and psychologically. Thus, some examples are more privileged than others, such as the prototypical examples (Hershkowitz, 1989). These typically form the subset with the most attributes, as they include the necessary attributes for the concept and additional more specific ones, sometimes with "strong visual characteristics" that can later act as distractions (Hershkowitz, 1990). According to Hershkowitz (1989), this is the prototypical phenomenon in which identification is influenced by visual perception.

Vinner and Hershkowitz (1983) identified three main reasons for certain types of behaviors in geometry tasks, which were later systematized as types of judgments that determine the expansion of the concept image beyond the prototypical example (Hershkowitz, 1989). In the first two types of judgment, the prototypical example is the representative of the concept, the basis, and the model for judging other examples, and the third type of judgment is analytical (Hershkowitz, 1989), as follows:

(i) judgment type I: the prototypical example is used as a reference, and a visual judgment is applied to other examples;

(ii) judgment type II: the prototypical example is used as a reference, and a judgment is applied based on non-critical attributes, usually the attributes of the prototype; and

(iii) judgment type III: the critical attributes that constitute the definition of the concept are used as a reference, and an analytical judgment is applied based on these attributes.

The prototypical phenomenon is therefore a natural characteristic of the cognitive development of the concept image, as it is from the prototypical example that the concept image develops, based on visual or analytical judgments (Hershkowitz, 1990). According to Vinner and Hershkowitz (1980), in the past there was an assumption that during cognitive tasks students resort to definitions, so there would be no need to present them with different examples of the concept. Nowadays, there is an acknowledgement that students resort to the concept image, and even when the definitions are known, prototypical examples may pose them obstacles in understanding the inclusion relationships between quadrilaterals (Fujita, 2012; Fujita & Jones, 2007; Ulger & Broutin, 2017).

Thus, it is important for students to have contact not only with prototypical examples but also with different examples and non-examples of the concept, in order to contribute to the formation of the concept image (Vinner, 1991). Similarly, it is necessary to overcome prototypical judgments, as they impose limitations by not allowing a clear distinction between examples and non-examples of concepts, unlike what analytical judgment favor (Hershkowitz, 1989).

In Brunheira and Ponte's (2018) study, for example, as the kite was the only quadrilateral whose prototypical model did not have parallel sides, the participating pre-service elementary teachers generally considered this attribute to define it, that is, a quadrilateral without parallel sides. In a subsequent investigation (Brunheira & Ponte, 2019), in the hierarchical classification of prisms conducted after a hierarchical classification of quadrilaterals, the pre-service teachers were more analytical. According to the authors, this can be explained by the fact that they may have learned about the classification process from one moment to another, and the limited familiarity of these pre-service teachers with prisms may have put prototypical attributes in the background, unlike during the classification of quadrilaterals. This result is in line with Naftaliev and Hershkowitz (2021), who concluded that the prototypical effect did not manifest when pre-service high school mathematics teachers, in a dialectical learning environment, developed their own examples and definitions for a geometric object they had not known until then.

1. Build each of the following quadrilaterals in the GeoGebra software: rectangle, parallelogram, rhombus, trapezium, isosceles trapezium, kite, square, and right trapezium.
2. Identify the minimum characteristics that define each of these types of quadrilaterals.
3. Find subgroups of quadrilaterals with common characteristics.
4. Construct a scheme with these types of quadrilaterals that highlights the relationships between them.
5. For each type of quadrilateral constructed, indicate the properties of its diagonals.

Figure 1. Task proposed to PMTs (Source: Research data)

Therefore, students should be actively involved in the processes of defining and classifying, specifically in geometry (de Villiers, 1994), and for this, it is necessary that teachers and pre-service teachers understand such processes.

CONTEXT AND METHODS

This study was conducted within a methods course, offered in the first semester of the second year of a Master for Mathematics Teaching at a university in Portugal. The class consisted of ten PMTs (who will be qualified to teach students aged 12 to 17), but the study includes only the seven PMTs who attended all classes related to the topic addressed in this research (PMTs are identified as PMT1, PMT2, ..., PMT7).

Two sessions of the course, each lasting two hours, were dedicated to the study of issues related to quadrilaterals and their teaching, during which the first two authors of this article acted as teacher educators (identified in the results as TE1 and TE2). In the first session, the PMTs were given a task to be solved with the GeoGebra software (**Figure 1**) and instructed to solve it in pairs or small groups. Due to PMTs' difficulties in the use of GeoGebra to make the construction and the limited time available for the session, the PMTs only completed the first item of the task, that is, they constructed the requested quadrilaterals in GeoGebra with the teacher educators' support.

In the second session, that is the focus of the present study, in pairs or small groups, the PMTs answered items 2, 3, and 4 of the task, and then a discussion followed focused on their resolution of these items and the systematization of the discussed ideas, according with a perspective of inquiry-based teaching (Oliveira & Cyrino, 2013; Rodrigues et al., 2022). From this discussion and systematization, we extracted the episode 1 that will be presented in the results. Subsequently, the PMTs were invited to respond to item

5 of the task, but due to the limited remaining class time, the responses were constructed and systematized through whole-class discussions between the teacher educators and the PMTs (episode 2).

This study follows a qualitative approach (Creswell & Poth, 2018), with the gathering of data from direct observations by the researchers, video recordings of the classes, and the written productions of the PMTs from the proposed task. The analysis focused interpretatively on the theoretical framework considered for the processes of definition and classification in geometry and the search for emerging characteristics of these processes in each of the two above mentioned episodes. For this purpose, the recording of the second session, in which the discussions and systematization of the mathematical task took place, was transcribed to portray the whole-class discussions. Subsequently, after several readings of the transcription the discussions emerged as constituents of a whole, as we observed that the construction of definitions and hierarchical classification resulted from interactions among the PMTs and between them and the teacher educators and were depended on the topic under discussion at each moment.

During the readings, we sought to identify characteristics of each of the two episodes, namely, those regarding the processes of constructing the quadrilaterals' hierarchical classification and the identification of the properties of the quadrilaterals' diagonals. Subsequently, we looked for information in the written productions of the PMTs in order to corroborate or refute our findings obtained from the analysis of whole-class discussions. The results are organized based on the two episodes, with excerpts from the whole-class discussions and some written records (WR) of the PMTs to reinforce the results derived from these collective discussions.

RESULTS

In the next two subsections, we present our results organized in the two episodes, namely: defining to construct a hierarchical classification of quadrilaterals; and identifying properties of quadrilateral diagonals from a hierarchical classification.

Episode 1: Defining to Construct a Hierarchical Classification of Quadrilaterals

During the whole-class discussion, the position of the kite in the hierarchical classification and its relationships with the other quadrilaterals, as well as its necessary critical attributes, were the focus of the first questions raised by the PMTs:

1. PMT5: We were wondering whether it was enough to say [about the kite] that two pairs of adjacent sides are congruent, or if we needed to add that the diagonals [of the kite] are perpendicular.

2. TE2: … The PMT3 and PMT4's group made an interesting point.

3. PMT4: Yes, I was saying that the rhombus is a special case of the kite.

4. PMT3: I disagree because the rhombus has its sides parallel in pairs and the kite does not.

5. PMT4: But by that definition from the other group [two pairs of adjacent sides congruent], it would work.

6. PMT6: … The rhombus is a trapezium, but the kite is not a trapezium.

7. PMT3: Yes! That's my point! … When I made [the scheme for] the quadrilaterals, I categorized them into trapeziums and non-trapeziums. The kite is in the non-trapeziums [subset] and the rhombus is in the trapeziums [subset], so I can't put them together. I think that to define a kite, there must be something else than just the adjacent sides being congruent.

When PMT4 raises the possibility of the rhombus being a special case of the kite (line 3), based on the attributes previously stated by PMT5 (type III judgment), this acts as a cognitive conflict factor for the rhombus and kite concept images of PMT3 (that is, a quadrilateral classified as a trapezium cannot simultaneously be a subset of a non-trapezium). For PMT3, the rhombus cannot be classified as a kite because there is a contradiction between the critical attributes of these two quadrilaterals (line 4). In this case, the PMT's judgment, even if based on properties, may have been based on a prototypical model of a kite, in which it does not necessarily have parallel sides (type II judgment). This fact can also be observed in the PMT6's response to item 2 of the task, which asked to identify the minimum characteristics of each of the quadrilaterals, when she says that the kite "does not have parallel sides and [has] two pairs of congruent adjacent sides" (PMT6, WR).

This conflict also occurs, most probably, because some PMTs established a dichotomous comparison for a classification in geometry, such as being a trapezium or not being a trapezium (line 7), or furthermore, for example, being a parallelogram or not being a parallelogram, as portrayed by PMT5 when answering question 4 of the written task. These facts limited them from considering other classification possibilities.

The perceived need not to classify the rhombus as a kite acted as a stimulus for the PMT3 to include attributes in her evoked concept image for this situation that went beyond those articulated by her, perhaps as a

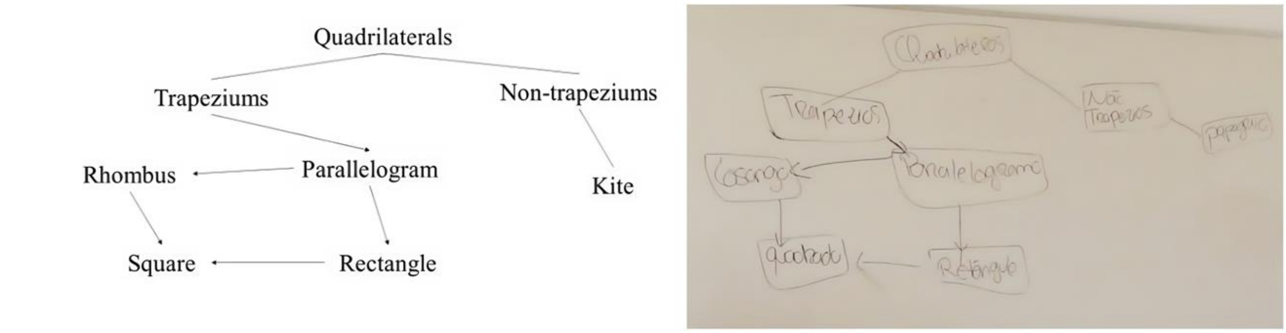

Figure 2. Scheme constructed on the board by PMT3 (translation and original) (Source: Research data)

memorized definition (line 9). Thus, PMT3, for example, raises the need to incorporate into her personal concept definition attributes (non-critical) of the kite that exclude the rhombus (line 7) and attributes (non-critical) of the rhombus that do not allow it to be classified as a kite (line 11) (type II judgment):

8. TE1: What is a rhombus? What are its minimal characteristics?

9. PMT3: Four congruent sides!

10. TE1: Is it sufficient to mention the four congruent sides?

11. PMT3: No, because that would overlap with the kite! For my definition of trapezium and non-trapezium, it can't … That's why I think to define a rhombus we have to say that the sides are parallel in pairs, because otherwise, it could be a kite.

Next, TE2 invites PMT3 to draw on the board her hierarchical classification scheme (**Figure 2**), as well as to justify to her colleagues the relationships she has considered.

12. PMT3: Therefore, trapeziums have one pair of parallel sides, parallelograms have two pairs of parallel sides. If they have all sides equal, they are rhombuses, if they have opposite sides equal and the others different, they are rectangles … So, from the parallelogram, two arrows come out and then both are connected to the square, because the square is both a rhombus and a rectangle. [The square] has right angles and all sides equal.

By mentioning that if the parallelogram has "all sides equal" it will be a rhombus, and if it has "opposite sides equal and the others different", it will be a rectangle, PMT3 likely made a judgment based on prototypical visual characteristics (type I judgment). Furthermore, her evoked concept image is influenced by a dichotomous comparison mobilized in the classification. Thus, as the rectangle was being compared with the rhombus, whose critical attribute is the congruence of the sides, this becomes the evoked attribute when she

Rectangle: right angles, parallel sides in pairs, opposite sides of equal length, ⊥ sides with ≠ lengths.

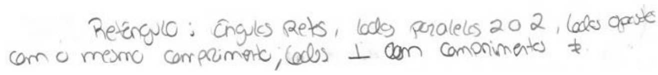

Figure 3. PMT3's answer to item 2 (translation followed by the WR) (Source: Research data)

describes the rectangle, coupled with a prototypical concept image of a rectangle that considers its sides' length. At that moment, PMT3 does not state the right angles as a rectangle's critical attribute, even though she evoked it at other times, such as in the written response to the item 2 of the task (**Figure 3**), when she was not establishing comparisons between quadrilaterals yet.

Similarly to PMT3, PMT1's personal concept definition of rhombus and rectangle is based on visual judgments of prototypical models (type I judgment) and dichotomous comparisons (line 13 in the following dialogue), that is, if one quadrilateral is classified by the congruence of its sides, the other, which is being compared to it at that moment, is classified by the non-congruence of its sides. The act of comparing two objects to classify them acted as a stimulus for the evoked concept image and personal definition, at that moment, to establish a dichotomous idea between the two quadrilaterals (rhombus and rectangle). Additionally, PMT1 also seems to state the critical attributes of the rectangle (line 16), not relating to his previous statement (line 13):

13. PMT1: … We have the class of equal sides and the class of different sides.

14. TE2: How do I define a rectangle?

15. PMT3: It can't be just by the [measurements of its] sides.

16. PMT1: The rectangle has perpendicular sides in pairs.

17. PMT3: That's a square!

18. PMT1: No! They are perpendicular in pairs, they don't have to [the measurements of the sides] all be equal.

19. PMT3: But if I have a square, I have perpendicular sides in pairs!

20. PMT1: Of course! The square is a subset of the rectangle.

Contrary to the excerpt above (lines 13 to 20), when stating the attributes of the square (line 12), PMT3 does not observe inconsistencies with the fact that the rectangle also has right angles. At that moment, it is possible that the length of the sides was the attribute she was evoking to define a rectangle and thus constituting a potential conflict. However, it becomes a cognitive conflict as two inconsistent pieces of information are simultaneously presented about the square and the rectangle. On one hand, she states that the square "has right angles and all sides equal" (line 12), on the other hand, she is now confronted with PMT1's statement that rectangles have "perpendicular sides in pairs" (line 16), and such attribute was not applied to the rectangle at that moment. However, prior to the discussion, when the PMTs were asked to describe the minimum and common characteristics of quadrilaterals, PMT3 wrote down "right angles" as a characteristic of both the rectangle and the square.

The critical attributes of the rhombus and the rectangle are stated in the discussion only when TE2 emphasizes the relationship they have with the square, that is, when questioning the PMTs about which attributes would be necessary to consider for the rhombus and the rectangle so that both "yielded the square". Faced with this question, PMT2 answers: "the rhombus [has] four congruent sides, the rectangle [has] four right angles". Thus, in this case, when comparing the attributes of the rhombus and the rectangle in relation to the square, the square's definition may have acted as a stimulus, leading PMT2 to evoke the right angles in his rectangle's concept image, that is, the necessary characteristic to classify the square as a rectangle.

To complete the construction of the scheme, the teacher educators questioned the addition of the kite into the hierarchical classification. PMT4 suggests, then, that it is enough to "remove the non-trapeziums [class] and directly defining the kite", that is, not to consider a classification based on trapeziums and non-trapeziums, in which the class of the kite would be a subset of non-trapeziums (as illustrated in **Figure 2**). With this, a new discussion begins about how to relate the kite to the other quadrilaterals based on the search for common critical attributes:

21. TE2: Therefore, what can we say about the kite?

22. PMT1: It has two equal opposite angles.

23. PMT2: No! Not opposite, adjacent sides.

24. PMT1: No! Opposite angles! Two of them are equal, the other two are either equal or different. If they are equal, it's a rhombus, and if they are different, it's a kite.

25. TE1: But is that a minimal characteristic? Is that how we define a kite?

26. PMT1: A quadrilateral with two equal opposite angles.

27. PMT2: [The kite] has both pairs of adjacent sides congruent.

28. PMT1: Ah, OK, exactly.

29. TE1: How can we fit the kite into the scheme? Does it stand alone?

30. PMT3: No! The rhombus is a kite … the rectangle can be a kite.

31. PMT1: So, the square is a kite!

32. TE2: So, the square is simultaneously …

33. PMT3: A rectangle, a rhombus, and a kite … the square is all of them!

PMT1 (line 24) describes a prototypical evoked concept image of a kite with characteristics aimed at contrasting dichotomously with the rhombus that is the quadrilateral to which it is being compared. Thus, if the rhombus has all angles congruent (also based on a prototypical concept image where the angles of the rhombus are all congruent), to characterize a kite it would suffice to say that the angles are not all congruent. It is noteworthy that this occurs even after the discussion about the difference between a rhombus and a rectangle, where the comparison attribute established by the PMTs was the length of the sides.

From this discussion with the PMTs and the previously constructed scheme (**Figure 2**), TE2 systematizes on the board the following final scheme (**Figure 4**).

Therefore, in the final scheme, the kite was not defined as a non-trapezium, allowing intersections with trapeziums. For instance, the rhombus and the square could be considered kites.

Episode 2: Identifying Properties of Quadrilateral Diagonals From a Hierarchical Classification

After having constructed the hierarchical classification scheme of quadrilaterals, the PMTs began answering item 5 of the task about the diagonals of

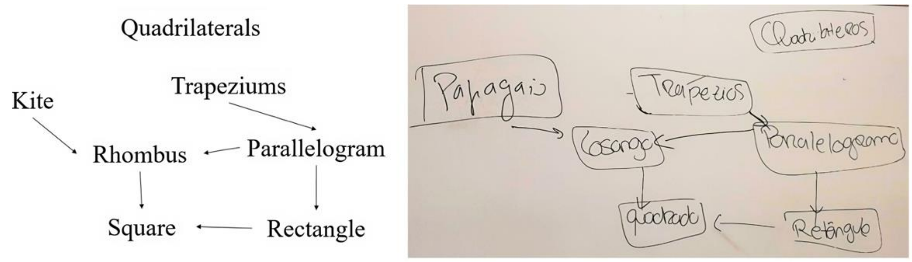

Figure 4. Final scheme of hierarchical classification (translation and original) (Source: Research data)

quadrilaterals. Due to the remaining short time, TE2 guided this part of the lesson so that the PMTs identified the characteristics of the diagonals and discussed them collectively, considering the established hierarchical classification (**Figure 4**).

34. TE2: Returning to the beginning, PMT2 and PMT5 had mentioned that the kite has perpendicular diagonals. So, what can we say from the hierarchical classification of quadrilaterals [about the diagonals of other quadrilaterals]?

35. PMT3: If the kite has perpendicular diagonals, then rhombuses and squares will also have them, because rhombuses are kites and squares are rhombuses.

In this case, PMT3 presents a global view of the situation, typical of hierarchical classification, with which colleagues seem to agree. Here, the definition and logical sequence of the situation place the use of prototypical attributes of quadrilaterals in the background, due to the established inclusion relationships.

However, at other times, the characteristic of the dichotomous comparison still persists due to the difficulty of some PMTs in establishing critical attributes and the still present, albeit less recurrent, prototypical phenomenon. PMT1, for example, compares the characteristics of the diagonals of kites and trapeziums without realizing that they cannot be contradictory (lines 36 and 38), characterizing a potential conflict:

36. PMT1: Teacher, but basically from the kite downwards [in the scheme] they have perpendicular diagonals, from the trapezium downwards they are not perpendicular.

37. TE2: But we have to say other things about the diagonals besides the angle. [For example] the lengths, if they are equal or not, if they bisect each other or not …

38. PMT1: Yes, okay. The lengths [of the diagonals] are always equal in trapeziums. It is in kites that they are not [equal]. In kites, [the diagonals] are perpendicular and of different lengths, and in trapeziums, they are non-perpendicular and of equal lengths.

39. PMT5: But what about squares?

PMT5 imposes a cognitive conflict factor in the situation (line 39) by questioning PMT1 about a possible contradiction in his speech. Subsequently, in a similar situation, PMT1 again elaborates contradictory logical relations due to the dichotomous comparison he establishes, this time, between the diagonals of the rhombus and the rectangle (line 40).

40. PMT1: The diagonals are perpendicular in the rhombus, and in the rectangle, they are not.

41. PMT4: In the rectangle, they are congruent.

42. PMT1: They are equal in length, yes.

43. PMT4: In the rectangle, the diagonals will be equal in length, and in the rhombus, the diagonals will be perpendicular.

44. TE1: … And in the square, what happens?

45. PMT1: The diagonals are not only congruent, but also perpendicular.

With regard to the square, possibly both because it is a familiar quadrilateral and due to the need to fulfill an inclusion relationship, PMT1 establishes logical relationships by deducing properties of the diagonals of the rectangle and the rhombus (line 45) (type III judgment).

Finally, the discussions of the PMTs seek to relate the attributes of the diagonals of the kite with those of the other quadrilaterals:

46. TE1: And here [referring to the kite], what happens [with the diagonals]?

47. PMT1: They are perpendicular.

48. TE1: Do they bisect each other?

49. PMT1: No! They intersect each other.

50. PMT2: Although the longer diagonal bisects the shorter diagonal.

> 51. TE2: Very well, there's a particular characteristic here, isn't there? So, will this characteristic occur for the rhombus and the square?
>
> 52. PMT2: No, the minimum characteristic of the kite that applies to all is that the diagonals are perpendicular.
>
> 53. PMT3: What PMT2 said about the longer diagonal crossing the midpoint of the shorter diagonal in the kite will happen in the case of the rhombus and the square because in them the diagonals bisect each other, so they will cross at the midpoint of the diagonals...
>
> 54. PMT1: Because the longer [diagonal] is the shorter [diagonal] at the same time.

In this case, the relation established by PMT3, and later complemented by PMT1, among attributes of the diagonals of the kite, the rhombus, and the square, was obtained through logical deduction, taking into account the properties of the quadrilaterals and their inclusion relationships in the hierarchical classification that should be respected (type III judgment).

DISCUSSION AND CONCLUSIONS

From the investigation of characteristics of the processes of constructing a hierarchical classification of quadrilaterals and identifying properties of quadrilateral diagonals with PMTs, we highlight below some points associated with the definitions and hierarchical relationships constructed by the PMTs, as well as their difficulties. We also discuss some implications for teacher education that can be derived from the results of this study.

The definitions were constructed (or at least systematized) by the PMTs during or for the construction of the classification, based on their concept image for each quadrilateral and their interactions with each other. This occurred because defining and classifying are interdependent processes (de Villiers et al., 2009), and the PMTs either did not recall the definitions of some quadrilaterals or had difficulties listing only the critical attributes when responding to the written task prior to the discussion.

One of the reasons for this difficulty in identifying critical attributes of quadrilaterals to construct definitions and hierarchical relationships is that, generally, the PMTs evoked prototypical concept images for quadrilaterals. Thus, they included unnecessary attributes to define certain classes of quadrilaterals or removed attributes to mistakenly include quadrilaterals in relationships they deemed appropriate, which reduced the list of common attributes among the quadrilaterals, making it difficult to establish relationships and hierarchical classification. This difficulty in identifying critical attributes to define a quadrilateral corroborates other results in the literature (Avcu, 2023; Miller, 2018; Zazkin & Leikin, 2008) and is generally associated with prototypical examples that have an extensive list of attributes, not all of which are necessary for the class (Brunheira & Ponte, 2019; Fujita, 2012).

However, we emphasize that, beyond a prototypical judgment of the quadrilateral itself, the recognition of critical attributes was also influenced by the characteristics, mainly prototypical, of the quadrilateral to which it was compared. Depending on the moment, the PMTs presented different critical attributes for the same quadrilateral, that is, they verbalized a personal concept definition based on the characteristics under discussion. Even though comparison is a process of mathematical reasoning associated with classification (Jeannotte & Kieran, 2017), the PMTs, most of the time, used it to distinguish two objects dichotomously, that is, when comparing two classes, they tended to compare them dichotomously.

In this sense, the construction of the hierarchical classification of quadrilaterals, with the elaboration of the scheme, was characterized by the construction or systematization of definitions to establish inclusion relationships; prototypical judgments; and dichotomous comparison, that is, in the comparison between two quadrilaterals, the attributes of one acted as a stimulus to activate certain attributes of the other, triggering dichotomous contrasts.

The PMTs identified the properties of the quadrilateral diagonals based on the hierarchical classification already constructed. This occurred because, in general, the PMTs did not recall the properties of the diagonals and then began to infer them from the hierarchical relationships they had built. This may have occurred because the properties of quadrilateral diagonals are not a commonly evoked critical attribute, as they are not often present in visual representations of these concepts. Thus, the diagonals may not be part of the concept image of quadrilaterals initially, requiring recourse to definition and logical deduction from the previously established hierarchical classification to identify their properties, which, to some extent, conditioned them to the inclusion relationships.

Unlike quadrilaterals, where the names of the concepts may suggest certain properties (for example, the word "rectangle" may suggest the property of right angles) and thus facilitate the identification of attributes (Avcu, 2023), this does not occur with diagonals. Furthermore, analogous to the results of other studies (Brunheira & Ponte, 2019; Naftaliev & Hershkowitz, 2021), the fact that the properties of quadrilaterals' diagonals were not so familiar to the PMTs may have contributed to more analytical judgments, with less influence from prototypical effects. However, the relationships established were still permeated, at times, by dichotomous comparisons that, beyond prototypical characteristics of quadrilaterals, considered their arrangements in the hierarchical classification. Thus, the identification of properties of quadrilateral diagonals was characterized by more influence from definition and logical relationships, both in judgments and comparisons, at the expense of prototypical judgments.

The proposal to construct a schema for hierarchical classification can potentially mitigate the difficulties faced in hierarchically defining and classifying, as demonstrated by Haj Yahya et al. (2024). Additionally, we observed that including the classes of trapeziums and kites in the schema's construction and conducting the study with PMTs in a collective environment generated cognitive conflicts and difficulties in elaborating the structure of the schema. For example, a prototypical judgment may have been one of the reasons why some of the PMTs initially classified quadrilaterals in two subsets: trapeziums (quadrilaterals with at least one pair of parallel sides) and non-trapeziums (quadrilaterals that do not have parallel sides) to include the kite. This is a result similar to the one reported by Brunheira and Ponte (2018), except for the dichotomous comparison characteristic present in our study. However, these conflicts also led the PMTs to discuss and reach conclusions, allowing them to identify critical attributes not only of the quadrilaterals involved in the conflict but also to clarify relationships with other quadrilaterals and how they could connect them in hierarchical relationships when constructing the schema.

In general, analytical judgments acted both as a stimulus to create cognitive conflicts and to resolve them. Thus, the use of definition and deductive logical relationships of inclusion generated cognitive conflicts in situations where concept images were evoked based on type I or type II judgments or by dichotomous comparison. On the other hand, it also assisted the PMTs in reaching conclusions and resolving conflicts. Therefore, the analytical judgment was an enhancing factor in hierarchical classification.

In the literature, it is already known that definition has the potential to "rescue from traps" set by the concept image (Vinner, 1991). However, a cognitive conflict arises only when two contradictory concept images are evoked simultaneously (Tall & Vinner, 1981). In our study, the interactions among the PMTs and the questioning from the teacher educators allowed for the generation of cognitive conflicts, the emergence of doubts, and subsequently, the consensus on what could be considered mathematically correct, through an iterative process of back and forward. Teacher education contexts in which PMTs can participate and raise questions and ideas are important for their education and future practice in various aspects, including the development of mathematical concepts (Tashtoush et al., 2022). The fact that the PMTs discussed possible definitions among themselves to establish certain classifications may have acted as moments of mutual explanation of the involved concepts, which positively influences classification (Alcock & Simpson, 2017). Thus, beyond the potential of definitions being constructed by the students themselves rather than merely presented to them (de Villiers et al., 2009), the identification of properties and the construction of definitions and hierarchical classifications of quadrilaterals can be enhanced when done collectively.

Therefore, we observed that the fact that the hierarchical classification scheme was constructed by the PMTs influenced and was decisive for them to identify properties of the diagonals later on. Similarly, elaborating the hierarchical classification scheme collectively was important for constructing and systematizing definitions of quadrilaterals, given the difficulties that the PMTs faced with prototypical phenomena and dichotomous comparison, to identify critical attributes and to establish inclusion relationships. The definitions and hierarchical relationships allowed for analytical judgments that drove the generation of cognitive conflicts and doubts, but through discussions, also led to conclusions and systematizations.

Implications, Limitations, and Future Perspectives

The experience investigated in this article, albeit challenging for the PMTs, is important to be developed in teacher education contexts so that they can experience and reflect on the difficulties, but also the potentialities, of a process of collectively constructing a hierarchical classification of quadrilaterals. Additionally, PMTs have the opportunity to reflect on different ways of defining and classifying quadrilaterals, which can also help them manage the diverse ways of thinking of their (future) students, as well as critically analyze curricula and textbooks.

This study also contributes to highlighting the role that processes of hierarchically defining and classifying play in geometric reasoning, as they require: identifying minimal defining characteristics of geometric objects; establishing geometric properties beyond visual judgments; analyzing and recognizing properties of different objects deductively; and logically relating and organizing sets of geometric objects.

Some points that we consider particularities of this study can, on the other hand, be seen as limitations. Firstly, since definitions are arbitrary, if we had considered other definitions, more or less inclusive, for the quadrilaterals in the study, the episodes could have taken on other characteristics. Similarly, discussing the critical attributes to define each quadrilateral before constructing the classification could have brought out other aspects. Finally, the results are specific to the number of PMTs who participated in the study and their contexts and previous experiences.

We consider, therefore, that some questions can still be investigated in order to clarify points that emerged from this research or to overcome its limitations. A pertinent aspect that was not addressed in this study relates to identifying possible relationships between the properties used and the knowledge mobilized by the PMTs in the GeoGebra construction, and the definition and classification they presented in the written task or in the whole-class discussion. Additionally, constructing hierarchical classifications based on the properties of the diagonals of quadrilaterals or other attributes, such as angles and axes of symmetry, may reveal different characteristics of hierarchical classification and its relationship with the process of defining, besides being a potential option to overcome the challenges of the prototypical phenomenon. We also observed that the teacher educators played an important role in provoking the PMTs with questions that in some instances generated cognitive conflicts, however that was not our focus of analysis in this study. Thus, future investigations could delve into such questions.

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