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Modeling the understanding of the vector concept by a Bayesian multidimensional item response model

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Abstract

In this paper, we propose to use a Bayesian three-dimensional item response model to estimate the student's understanding of the vector concept. The experiment involved administering a test with 20 items about understanding vectors to 120 undergraduate students. The understanding is considered a latent variable of three dimensions related to observable data from item responses of a vector test. The Bayesian approach was used to obtain estimations about individual parameters, which were qualitatively analyzed based on a framework for understanding concepts. According to the results, we classify students into three levels of understanding in each dimension analyzed. We observed that a high percentage of students reached a medium level of understanding, while a low percentage achieved a high level of understanding. In addition to the classification, we obtained understanding profiles to quantify the level of students' understanding in each of the dimensions. These profiles offer a more nuanced view of students' understanding of the concept vector, which has significant practical implications. This information can guide the improvement of teaching strategies and curriculum design, allowing educators to address specific areas of difficulty and enhance learning in the concept of vectors.

Keywords: understanding, vector concept, multidimensional item response model, undergraduate students

INTRODUCTION

The vector is a very important concept in mathematics and physics, since it is essential to study topics of analytical geometry, linear algebra, vector calculus, vector analysis, among others (Gacovska-Barandovska et al., 2020; Harel, 2021; Kösa & Karakus, 2010; Parraguez & Oktaç, 2010; Stewart & Thomas, 2009), and for modeling real-world phenomena and characterizing physical magnitudes having magnitude and direction; for example, motion, force, velocity, acceleration, magnetic field, electric field. These magnitudes require knowledge of the graphic and algebraic representations of vectors and their basic operations, such as addition, subtraction, dot product, cross product (Barniol & Zavala, 2014b; Heckler & Scaife, 2015; Knight, 1995; Mauk & Hingley, 2005; Mikula & Heckler, 2017; Scaife & Heckler, 2010; Shodiqin & Taqwa, 2021).

Many authors have found that the vector concept is a difficult topic to learn for students, mainly in adding and subtracting vectors graphically, determining direction, calculating magnitude, operating graphically with vectors, performing basic operations using unit vectors, determining components, calculating the dot product, and calculating the cross product, among others (Angateeah et al., 2017; Barniol & Zavala, 2014a, 2014c, 2016; Carli et al., 2020; Donevska-Todorova, 2015; Flores-García et al., 2008; Latifa et al., 2021; Özdemir & Çoramik, 2018; Qonita & Ermawati, 2020; Tairab et al., 2020). These difficulties are due to the lack of understanding of the vector concept (Knight, 1995; Flores-García et al., 2007; Nguyen & Meltzer, 2003; Wutchana, 2021). A good understanding can amend difficulties and enable students to learn other related or advanced concepts. On the other hand, a poor understanding of the vector concept significantly affects the correct interpretation and application of principles in

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Contribution to the literature

- In this study, we propose to use a three-dimensional item response theory (IRT) model was used to estimate undergraduate students' understanding of vector concepts. According to the understanding estimates obtained, we classify students, for each dimension, into low, medium and high understanding about the vector concept.
- We developed an understanding profile for each student, which represents the pattern of mastery in the comprehension dimensions analyzed. The results of this research reveal that most students achieved a medium level of understanding of vectors.
- Understanding profiles are graphically displayed, allowing teachers or researchers to easily identify students' strengths and weaknesses in each dimension of understanding.

various areas of knowledge. For example; in physics, there is confusion between magnitudes or mistakes in the interpretation of forces; in chemistry, scalar values are confused with vectors. Then, is it possible to quantitatively know a student's level of understanding? What tools or methods are most effective for measuring understanding of the vector concept in students at different educational levels? Therefore, our interest in this work is to estimate the level of students' understanding of this concept by using a Bayesian item response model.

In previous research, understanding of vectors has been studied: for example, through multiple-choice tests, Barniol and Zavala (2014d) assessed the understanding of the vectorial representation and informed that students have difficulty understanding concepts such as unit vectors, dot product, and cross product. Sirait and Oktavianty (2017) studied the ability to understand vectors (addition, subtraction, and components) and concluded that addition and subtraction in two dimensions were more challenging to understand than vector components because most students could determine the direction and magnitude of vector components and identify x, y components. Latifa et al. (2021) identified students' difficulties understanding vector concepts, and they adopted five categories of understanding: very less, less, adequate, good, and very good. The results indicated that students have on average, less or even, very less understanding, and their main difficulty is representing vectors graphically. Taqwa and Rahim (2022) compared students' abilities to understand the addition and subtraction of vectors in one, two, and three dimensions using visual (graphic) and mathematical (symbolic) representations. The results showed that the students' ability to understand vector concepts with mathematical representations was better than visual representations since students are more accustomed to operating vector algebra.

Regarding understanding levels, Saraçoğlu and Kol (2018) investigated students' level of understanding and misconceptions about vector magnitude, addition and subtraction of vectors, dot product, and cross product. They classified the responses into four understanding levels: full understanding, partial understanding, misunderstanding, and not understanding. They conclude that, in general, the students showed a partial understanding of vector magnitude, a full understanding of the addition and subtraction of vectors, a misunderstanding of the vector dot product, and no understanding of the vector cross product.

Alam (2020) studied the understanding by assessing students' cognition about vectors (magnitude, direction, addition, subtraction, components, dot product, and cross product). He expressed the results in percentages according to indicators, such as correct, partially correct, wrong, and blank (an option for the problems the students could not answer), and described the student's successes and mistakes qualitatively. He found that most students who begin university need an adequate understanding of vectors.

In this paper, to carry out our objective, we define the understanding as a latent variable that cannot be measured directly because it represents a hypothetical construct such as intelligence or motivation (Torgerson, 1958); however, it is possible to use other manifest variables to know it, such as the item responses which serve as indicators to measure the underlying construct (Bollen, 2002; Valdés et al., 2019). Furthermore, we consider that understanding is not a one-dimensional variable but rather that it is built by three dimensions or characteristics, which have to do with the definition, distinction, and application of the vector concept. To analyze this task, we use a multidimensional item response theory (MIRT) model, which defines a mathematical relationship between student's item responses and a latent variable of three dimensions that represents the understanding level that the items measure (Fox, 2010) from a Bayesian approach of the statistic.

Many educational systems and assessment standards are deeply rooted in traditional methodologies, which limits innovation and the adoption of more advanced approaches such as the Bayesian MIRT models. However, the Bayesian MIRT models offer the advantage of incorporating prior information about students' abilities, which improves the estimation of understanding, as in the context of this paper. Furthermore, ability estimates are more robust, especially in small samples, as they combine observed data with prior information. This approach also allows for obtaining credibility intervals instead of point estimates, thus facilitating the assessment of uncertainty in ability estimates.

CONCEPTUAL FRAMEWORK

Understanding Concepts

For education, it is essential how much students achieve in their learning, and this is associated with understanding, thought abilities, and knowledge, which must be promoted and assessed. Understanding concepts involve the ability to apply definitions, give examples and counterexamples, use symbols to present a concept, use different forms of representation of a concept, identify the characteristics of a concept, compare concepts, interpret a concept, apply a concept in the context that is required, and relate a concept with other concepts (Al-Mutawah et al., 2019; Gnaldi, 2017; Haji & Yumiati, 2019).

Trowbridge and McDermott (1980) argue that understanding develops from intuition, experience, and perception of previous instructions. They also consider that understanding can be assessed by

(1) defining a particular concept in an acceptable operational manner,

(2) distinguishing the concept from other related concepts, and

(3) applying it.

We considered these three aspects as dimensions of understanding in correspondence with the proposed MIRT model.

Understanding Dimensions

The dimensions to achieve the understanding of the vector concept are described, as follows:

Dimension 1. Define a particular concept in an acceptable operational manner

This dimension refers to knowing and using a particular concept's definition to perform operations correctly. In the case of vectors, the student must be able to carry out basic operations with this concept (addition, subtraction, multiplication by a scalar, dot product, and cross product) correctly.

Dimension 2. The distinction of the concept from other related concepts

The student must differentiate the concept from other related concepts. Mainly, it is possible to know if the individual distinguishes the characteristics of the vector concept and its properties. For instance, he must distinguish when and how to perform the vector cross product, realize the dot product, and distinguish the nature of their results.

Dimension 3. The application of the concept

For this dimension, the student must be able to apply the concept in different contexts. In this research, the students can solve exercises or problems in mathematical contexts and related disciplines using the vector concept in different representations (graphic, algebraic, and unit vectors). For this, it is necessary for the student to correctly apply the magnitude, direction, and sense of a vector.

Bayesian MIRT Model

The IRT models have been successfully employed in modern educational measurement and contribute significantly to research in education specifically in mathematics and physics education (Hori et al., 2020; Milovanović & Branovački, 2020; Rakkapao et al., 2016).

The MIRT model is a statistical model that describes the interactions between multiple students' abilities and the test items. In many cases, multiple abilities are required to perform successfully within a domain. This model can deliver more detailed information than information obtained from classical measurement models within the educational context; that is, the results can be presented in individual profiles for multiple abilities instead of providing one single score (Hartig & Höhler, 2009; Kunina-Habenicht et al., 2009; Min & Aryadoust, 2021). In this context, we model understanding of the students as a three-dimensional latent variable using a MIRT model. We described the MIRT model used next.

Considers that Y_{ik} is a random variable denoting the response of student i to item k, then the probability of the correct answer p_{ik}, corresponding to i-th student in the k-th item is given by Eq. (1) (Reckase, 2009):

$$P(Y_{ik} = 1|\boldsymbol{\theta}_i, \boldsymbol{a}_k, d_k) = p_{ik} = \frac{e^{\boldsymbol{a}_k\boldsymbol{\theta}'_i + d_k}}{1 + e^{\boldsymbol{a}_k\boldsymbol{\theta}'_i + d_k}}, i = 1, \dots, N;\ k = 1, \dots, K, \quad (1)$$

where $\boldsymbol{a}_k\boldsymbol{\theta}'_i + d_k = \sum_{l=1}^{3} a_{kl}\theta_{il} + d_k$, $\boldsymbol{\theta}_i$ is a vector of the parameter indicating the understanding of i-th student, $\boldsymbol{a}_k$ is a vector of item discrimination parameters of the k-th item, d is called the intercept parameter. Note that this model is a multidimensional extension of the two-parameter logistic model (2PL) and represents the conditional probability that the i-th student responds correctly to the k-th item given an understanding level $\boldsymbol{\theta}_i$.

Traditionally, frequentist analysis has been used in IRT; however, the Bayesian approach becomes very attractive for modeling item response data (Fox, 2010). So, we make statistical inferences from the Bayesian approach, in which the parameters of interest are considered random variables. In the framework of this paper, $\boldsymbol{\theta}$ is the parameter of interest, which represents

Table 1. Properties of the vector concept (adapted from Barniol & Zavala, 2014a)

Properties	Item	Description
Direction	5	Choosing a vector with the same direction from among several in a graph.
	17	Calculation of the direction of a vector written as unit vector notation.
Magnitude	20	Calculation of the magnitude of a vector written as unit vector notation.
Graphic representation	10	Graphic representation of a vector expressed as unit vector notation.
Component	4	Graphic representation of the y component of a vector.
	9	Graphic representation of the x component of a vector.
	14	Calculation of magnitude of the x component of a vector (angle measured from the y axis).
Unit vector	2	Graphic representation of a unit vector.
Addition	1	Graphical addition of vectors in 2D.
	7	Comparing the magnitude of the vector addition of two same magnitude vectors at 90° with the magnitude of the vectors.
	16	Comparing the magnitude of the vector addition of two same magnitude vectors at 143.13° with the magnitude of the vectors.
Subtraction	13	Graphical subtraction of vectors in 2D.
	19	Graphical subtraction of a vector in 1D.
Scalar multiplication	11	Graphic representation of a vector multiplied by a negative scalar.
Dot product	3	Geometric interpretation of dot product as a projection.
	6	Calculation of dot product using the equation ABcosθ.
	8	Calculation of dot product of vectors written in unit vector notation.
Cross product	12	Geometric interpretation of cross product as a perpendicular vector.
	15	Calculation of cross product of vectors written as unit vectors notation.
	18	Calculation of a cross product magnitude using the equation $ABsen\theta$.

the understanding level of a student about the vector concept and has a prior distribution beforehand; the responses to a pair of items are statistically independent given the understanding parameter. Let $\boldsymbol{Y} = (y_{ik}), i = 1, \dots, N, k = 1, \dots, K,$ denote the observed dichotomous response matrix. Then, the likelihood function is given byEq. (2):

$$L(\boldsymbol{\theta}|\boldsymbol{Y}) = \prod_{i=1}^{N} \prod_{k=1}^{K} p_{ik}{}^{y_{ik}} (1 - p_{ik})^{y_{ik}}, \quad (2)$$

where p_{ik} is defined in Eq. (1). From the Bayesian point of view and based on $\boldsymbol{Y}$, we are interested in the posterior distribution of $\boldsymbol{\theta}$**,** which is obtained through the Bayesian theorem, as follows (Eq. [3]):

$$P(\boldsymbol{\theta}|\boldsymbol{Y}) = \frac{L(\boldsymbol{\theta}|\boldsymbol{Y})\, P(\boldsymbol{\theta})}{P(\boldsymbol{Y})}, \quad (3)$$

where $P(\boldsymbol{\theta})$ is the prior distribution of $\boldsymbol{\theta}$ and $P(\boldsymbol{Y})$ es the marginal distribution of $\boldsymbol{Y}$. We use Markov chain Monte Carlo (MCMC) methods to obtain samples from Eq. (3) employing the JAGS software (Plummer, 2012), within R software (R Core Team, 2023) because $P(\boldsymbol{\theta}|\boldsymbol{Y})$ is analytically intractable.

METHODS

Participants

In this study, 120 Mexican engineering and mathematics students between 18 and 22 years old participated. The careers share subjects, for example, both differential and integral calculus, vector calculus, analytical geometry, and linear algebra. The participants were chosen under the criteria of having studied the vector concept and that they agreed to participate.

In addition, a simple random sampling of a population of students from a university in Guerrero was carried out, achieving a significant sample size with an estimation error of 3% and a confidence level of 95%. Since there was no prior information on the student's abilities concerning understanding the vector concept, they were all assigned the same non-informative prior information.

Instrument

For data collection, we used the *test of understanding of vectors* (TUV) designed and widely studied by Barniol and Zavala (2014a). The TUV is a multiple-choice test with 20 items, which are questions about ten properties of the vector concept, such as direction, magnitude, graphic representation, components, unit vector, addition, subtraction, multiplication by a scalar, dot product, and cross product (**Table 1**).

Barniol and Zavala (2014a) carried out the design of the TUV based on the dimensions of understanding proposed by Trowbridge and McDermott (1980). Each test item has five answer options, of which only one is correct. The remaining wrong answers characterize students' most common mistakes when working with the vector concept. Moreover, these authors provide evidence detailed on the validity, reliability and discriminatory power of the TUV using five statistical proofs: difficulty index, discrimination index, biserial point coefficient, Kuder Richardson, and Ferguson's delta. In other research, the characteristics of the TUV items have been studied through different IRT models; for example, Rakkapao et al. (2016) applied the three-

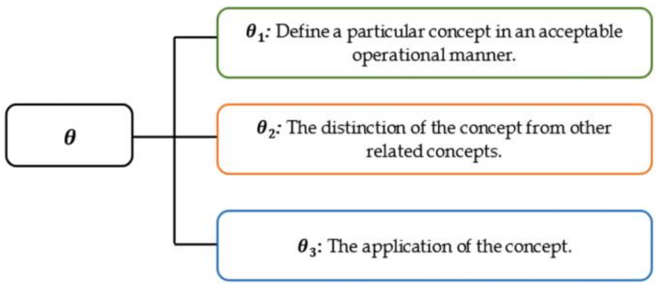

Figure 1. Dimensions to understand a concept (Source: Authors' own elaboration)

parameter logistic IRT model to analyze each item of the test and the item response curve technique to analyze the distractors of each item. Susac et al. (2018) corroborated the appropriate performance of the TUV items using the Rasch model. In contrast with previous research, we propose to use a three-dimensional IRT model to estimate students' understanding level of the vector concept.

Data Collection and Analysis

The response data were collected by applying the TUV. The students were not previously informed about the test application and were randomly selected. The test was applied in their classrooms for about 25 minutes, and they were requested not to leave items unanswered. We used the SN notation (S: student, N: student number) to identify the students.

To obtain the response matrix, we assign the value of 1 if the answer is correct and 0 otherwise, obtaining a data frame of 120 lines corresponding to the students and 20 columns corresponding to the items. To model the student's understanding level of the vector concept, we use Eq. (1) with three dimensions, $K = 20$, $N = 120$, and $\boldsymbol{a}_k\boldsymbol{\theta}'_i + d_k = a_{k1}\theta_{i1} + a_{k2}\theta_{i2} + a_{k3}\theta_{i3} + d_k$. The 2PL multidimensional item response model in Eq. (1) considers three dimensions of the parameter θ, corresponding to the three aspects to understand a concept proposed by Trowbridge and McDermott (1980). We consider these three characteristics to be abilities that a student requires to understand the vector concept (see **Figure 1**).

Bayesian estimation in the MIRT model produces reliable estimates even in the presence of variability and noise in the data. In mathematics, where abilities and concepts interrelate in complex ways, this robustness is essential, unlike classical IRT models. By considering multiple dimensions, the MIRT model allows a more detailed assessment that goes beyond aggregate scores, providing a more complete understanding of each student.

Since the joint posterior distribution is analytically intractable and the marginal posterior distributions of the parameters are complicated; then, we can obtain samples from Eq. (3) using MCMC techniques. The most common MCMC methods are the Gibbs sampling (Casella & George, 1992; Gelfand & Smith, 1990) and the Metropolis-Hastings (Chib & Greenberg, 1995; Metropolis et al., 1953). Currently, many of the MCMC algorithms have been already implemented in computer programs, such as WinBUGS and JAGS (Plummer, 2012), Stan and t-walk. All of these software packages provide programs for Bayesian modeling through posterior simulation given a specified model and data. The R packages, such as R2jags (Su & Yajima, 2015) and rjags (Plummer, 2015), allow one to run JAGS from within R software (R Core Team, 2023). In this paper, we use JAGS within R to obtain samples from the marginal posterior distributions of interest.

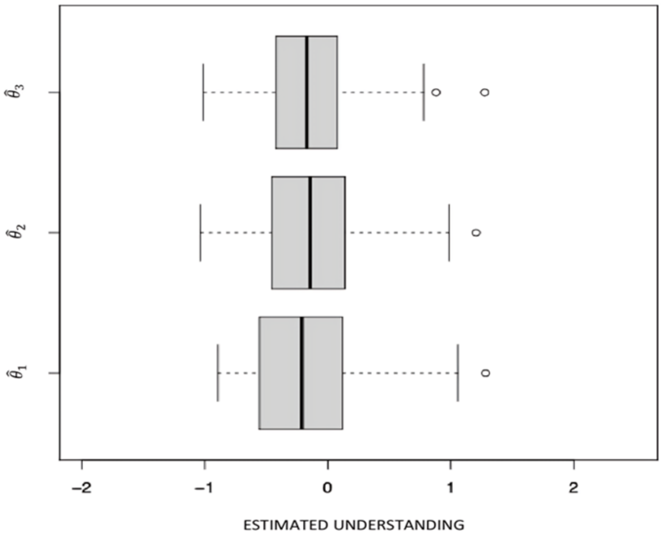

Figure 2. Estimated students' understanding level of each dimension (Source: Authors' own elaboration)

RESULTS AND DISCUSSION

Reliability of the Test

We analyze the TUV's reliability using two internal consistency measures: Cronbach's alpha and Kuder-Richardson. For the 20 items of the test, Cronbach's alpha was 0.705 and Kuder-Richardson 0.71, which indicates that the test applied has an acceptable reliability.

Students' Understanding Level

We present the results concerning the implemented model in Eq. (1). We obtain Bayesian point estimations of students' understanding level in each dimension denoted by $\hat{\theta}_{ij}, i = 1, 2, \ldots, 120, j = 1, 2, 3.$ These estimations represent 120 values of the understanding of each student in each dimension and are graphically presented in a boxplot (see **Figure 2**). According to the IRT model, the values range of the estimated understanding from -4 to 4, then zero is a reference point. In this context, values of $\hat{\theta}$ greater than zero mean that students understand the vector concept better; in contrast, negative values of $\hat{\theta}$ mean that students do not

Table 2. Rate of students in each category according to the estimated level of understanding

Dimension	Understanding	Interval	Rate of students
Define a particular concept in an acceptable operational manner (θ_1)	Overall	[-0.964, 1.267]	100% (120)
	Low-level	[-0.964, -0.35)	30% (36)
	Medium-level	(-0.35, 0.058)	45.8% (55)
	High-level	(0.058, 1.267]	24.2% (29)
The distinction of the concept from other related concepts (θ_2)	Overall	[-1.057, 1.279]	100% (120)
	Low-level	[-1.057, -0.3619)	37.5% (45)
	Medium-level	(-0.3619, 0.1025)	38.3% (46)
	High-level	(0.1025, 1.279]	24.2% (29)
The application of the concept (θ_3)	Overall	[-0.979, 1.243]	100% (120)
	Low-level	[-0.979, -0.3661)	28.3% (34)
	Medium-level	(-0.3661, 0.0441)	45% (54)
	High-level	(0.0441, 1.243]	26.7% (32)

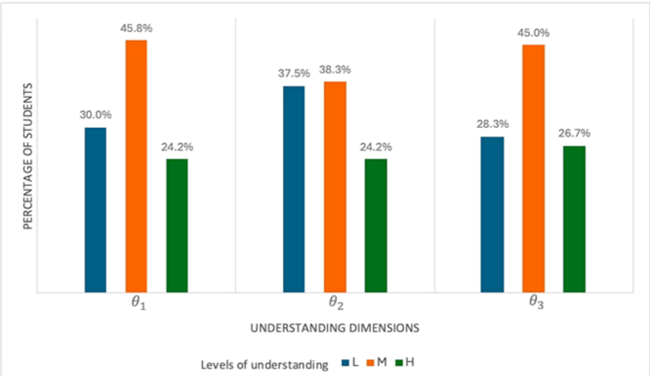

Figure 3. Percentage of students in each understanding level in each dimension (Source: Authors' own elaboration)

achieve a good understanding. The estimates obtained range from the understanding, shown in **Figure 2**, from -1 to 1.2 approximately. Based on these values and for interpretive purposes, we categorize students into three levels of understanding: low, medium, and high. Then, if a student has an understanding close value to -1, it means that he has a low understanding level; by contrast, if his estimated understanding level is close to 1.2, it means he has a high understanding. In order to obtain the student classification intervals according to their estimated understanding level, we use $\underline{\theta}_j \pm 0.5\, Sd(\hat{\theta}_j)$, where $\underline{\theta}_j$ is mean of the $\hat{\theta}_{ij}$ and $Sd(\hat{\theta}_j)$ is standard deviation of $\hat{\theta}_j$. In **Table 2**, we present the intervals obtained and the corresponding percentage/number of students in each category.

In **Figure 3**, we can see that in the three dimensions, the highest percentage of students is concentrated in the medium level of understanding, followed by the low and high levels.

The mean estimation in each understanding dimension is $\underline{\theta}_1 = -0.1472, \underline{\theta}_2 = -0.1299$, and $\underline{\theta}_3 = -0.1610$, which means that the students generally had a medium understanding level of the vector concept. These results support what is shown in **Figure 3**.

The MIRT model allowed us to develop understanding profiles of each student, from which more information can be obtained from the responses of the items instead of only correct or incorrect answers. In correspondence with this research, we call them understanding profiles (Hartig & Höhler, 2009; Wu & Adams, 2006). These profiles represent the pattern of students' domain in the three dimensions $\theta_1, \theta_2, \theta_3$.

Table 3. Types of understanding profiles

Profile type	Number of students
MMM	30
LLL	28
HHH	24
MLM	14
MHM	6
HMH	6
LML	5

Students' Understanding Profiles

Through understanding profiles, it is possible to know in detail the students' strengths or weaknesses in each dimension of understanding; for example, the number of students who dominate or have weaknesses in some of the dimensions of understanding, groups of students with equal understanding profiles or, the number of students who have a specific ability. Then, these profiles serve as a framework of reference in teaching.

To denote each student's understanding profile, we use the first letter of the level corresponding to them according to their estimate value $\hat{\theta}_{ij}$. For example, we denote the HMM profile by referring to a student who has a high level of understanding in dimension θ_1, and a medium level of understanding in dimensions θ_2 and θ_3. Hence, we consider 7 types of profiles (see **Table 3**) since more than 90% of the students fit them.

The predominant profile was MMM, with 30 students, which means a medium level of understanding in the three understanding dimensions. The second most frequent profile was the LLL; 28 students showed a low understanding level $\theta_1, \theta_2, \theta_3$. The third recurrent profile was the HHH, which indicates a high understanding of the operational definition, distinction, and application of the vector concept. Other profiles found were the MLM, MHM, HMH, and LML.

The MMM profile indicates that the three dimensions of understanding must be strengthened, and then it is necessary to analyze their weaknesses. For example, we observed that students with this profile had difficulty solving items 8, 11, and 12 since only one student answered correctly; in items 2, 15, and 19, two students answered correctly; in item 13, three students answered correctly; in items 16 and 17, four students answered correctly, and in item 20 five students answered correctly. Five items involve the unit vector notation, items 2, 8, 15, 17, and 20.

In item 2 (graphic representation of a unit vector), there were two common incorrect answers: option B (10 students) and option E (16 students). Option B implies a unit vector that has x and y components of one unit. Barniol and Zavala (2014a, 2014d) reported that students choosing option B, believe that this unit vector has a magnitude of 1; that is, students perform incorrectly the process of identifying the unit vector of a given vector (normalization process) or as Barniol and Zavala (2012) mention, some students simply do not operate. Susac et al. (2018) state that students did not correctly use the notion of vector magnitude and vector decomposition. Consequently, we consider that dimensions θ_1 and θ_3 must be favored. On the other hand, option E refers to the same given vector. Barniol and Zavala (2014d) found that the student's argument for this response is that the unit vector of a given vector is the addition of the two components written in the unit vector notation, which yields the same given vector. Then, in this case, it is also necessary to contribute dimension θ_2 because they do not distinguish the characteristics of the unit vector.

In item 8 (calculation of the dot product), we noted the two most frequent incorrect answers, option C (15 students) and option D (12 students). This result agrees with what was found by Barniol and Zavala (2014a, 2014d), who mention that students calculated the product of vector $1i$ (from vector $\vec{A}$) and vector $5i$ (from $\vec{B}$) obtaining $5i$, and then added $3j$ (from vector $\vec{A}$) to the result. For the above, we can associate with students incorrectly carrying out the dot product procedure, and do not identify that the dot product results in a scalar then, dimensions θ_1 and θ_2 must be improved. Option D is the result of adding $\vec{A}$ and $\vec{B}$, which indicates that students confuse the dot product with adding vectors. Therefore, it is necessary to improve the understanding in dimension θ_2.

In item 15 (calculation of cross product), there were two common incorrect answers: option C (10 students) and option E (10 students). In option C, students multiplied $1i$ (from $\vec{A}$) and $5i$ (from $\vec{B}$) obtaining $5i$, and added the vector $3j$ (from $\vec{A}$) to the result. And in option E, students added $1i$ (from $\vec{A}$) and $5i$ (from $\vec{B}$) obtaining the vector $6i$, and adding the vector $3j$ (from $\vec{A}$) to the result. In both cases, this denotes that the students incorrectly developed the cross-product. Therefore, dimension θ_1 must be amended. Likewise, in item 17, the correct answer is 126.87°, but there were two frequent incorrect answers: options B (11 students) and D (11 students). In option B determined the angle 53.13° as $\left(\frac{4}{3}\right)$, in option D measured the vector angle counterclockwise from the x-axis, and they assumed the vector is in the third quadrant at 45°. Hence, is necessary to favor dimension θ_3, because they applied the direction of the vector incorrectly. During the instruction, pointing out the measurement of angles is essential.

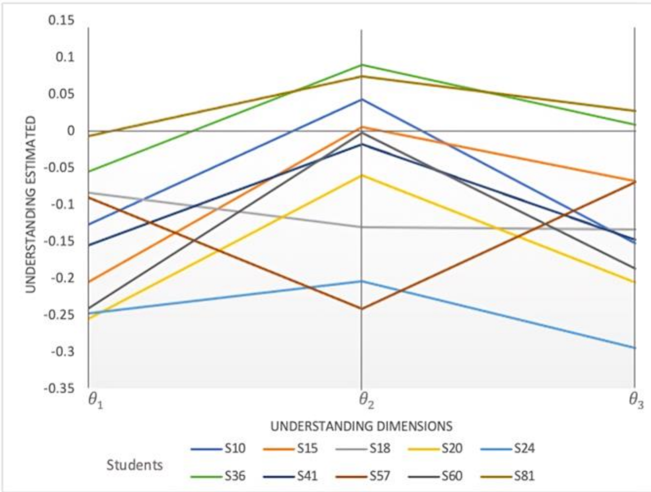

Figure 4. Example of students with MMM profile (Source: Authors' own elaboration)

In item 20, students failed to determine the magnitude of the vector $\vec{A} = 2i + 2j$, possibly they added $2 + 2 = 4$ or incorrectly applies the Pythagorean Theorem ($|\vec{A}| = \sqrt{2^2 + 2^2} = 4$). In the same case of item 17, is necessary to favor dimension θ_3, because they incorrectly applied the magnitude of the vector.

In general, we observed that students had difficulties in developing the dot product and cross product, multiplying a vector by a negative scalar, in the geometric interpretation of the cross product, subtracting two vectors in one dimension, and determining the magnitude of a vector; however, the responses reflected that the biggest obstacle was the representation of unit vectors. Research by Barniol and Zavala (2014c) mentions that a few studies about vector concepts in mathematical and physical contexts have focused on difficulties in problems that involve unit vector notation. This notation is important because it is commonly used in mathematics and physics university courses. Strengthening skills in the unit vector representation and procedures such as normalization of a vector, calculating angles, and the Pythagorean Theorem is essential for studying vectors. In **Figure 4**, we can graphically see some MMM understanding profiles.

An LLL understanding profile means a low understanding of the three dimensions analyzed in this

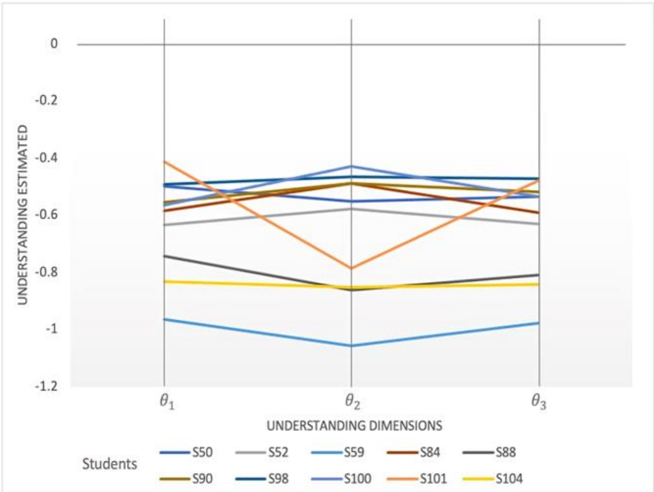

Figure 5. Examples of students with LLL profile (Source: Authors' own elaboration)

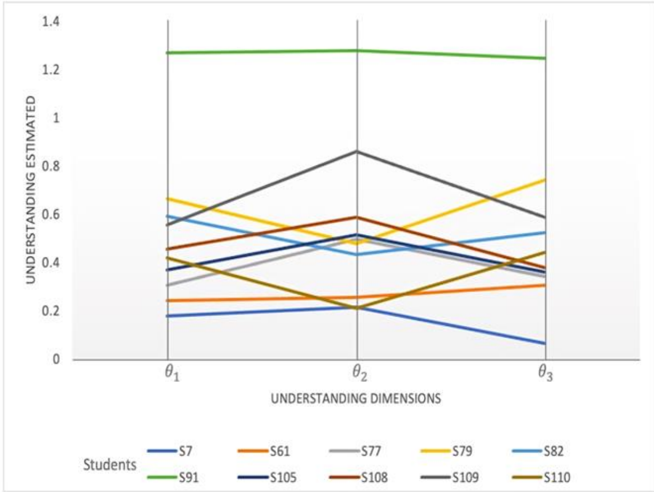

Figure 6. Examples of students with HHH profile (Source: Authors' own elaboration)

research, making it difficult to understand the vector concept well. The number of correct answers was between 0 and 4. There were zero correct answers in items 11, 13, 15, and 17; in item 20, there was one correct answer. Particularly, S59 failed in all test items, in **Figure 5** we can graphically the behavior of this student and other LLL understanding profiles.

These results may be due to the lack of prior knowledge about vectors and their operations or mathematical skills such as domain on angles, the trigonometric function's sine and cosine, the Theorem of Pythagoras, and the rest. We consider it critical to explore whether students have this previous knowledge before studying vectors. For example, in their research, Mikula and Heckler (2013) conceive vector math operations (vector addition, subtraction, components, tilted coordinate systems, dot products, and cross products in both algebraic and arrow representations) as essential skills because are simple procedurals that are required to solve different problems and are a fluent skill of experts. These skills involve applications and a simple conceptual understanding of basic calculation procedures, including basic math and interpretations of representations. Authors mention that some vector math skills must be broken further into subskills (e.g., finding components of vectors, trigonometric functions, angles, and others); however, they detected that even post instruction, students have difficulties in vector subskills, for example, often confuse sine with cosine when the angle is given from vertical and commit sign errors when the angle is given from the tip of the vector.

It has also been documented that students begin university studies with an inadequate understanding of vectors (Alam, 2020). In the same way, Nguyen and Meltzer (2003) state that approximately half of the undergraduate students do not have useful knowledge about vectors, even if they have taken a physics course. Vector operation difficulties have been observed in pre-college teachers and graduate students by Wutchana et al. (2015), who point out that this is due to teachers thinking work with some of the vector concepts is easy and they develop the topic naturally, causing some difficulties for the students. Similarly, Mikula and Heckler (2017) mention teachers believe that knowledge about vectors is a requisite for university courses or believe that it is sufficiently practiced in the current course, and they do not dedicate enough time to develop this topic. For their part, Mikula and Heckler (2013) consider that although vector math is an important topic and mathematics or physics textbooks include an early chapter on the topic, this is not sufficient.

A HHH profile means that the student understands the concept's operational definition, distinction, and application. Twenty-four students had this profile, for example, S7, S61, S77, S79, and S91, and obtained the highest understanding values (**Figure 6**). These students only needed help recognizing the geometric interpretation of the dot product (item 3) and identifying the mathematical expression for determining the magnitude of the dot product (item 6), because of this, dimension θ_2 must be strengthened.

It is convenient for this group of students to point out their strengths. They had a better performance on item 10 (graphic representation of a vector expressed as unit vector notation) 20 students answered correctly; in item 9 (graphic representation of the x component of a vector), 19 students answered correctly; in item 5 (choosing a vector in the same direction from $\vec{A}$) 18 students answered correctly, and in items 7 (calculation of magnitude of the addition of $\vec{A}+\vec{B}$) and 20 (calculation of the magnitude of a vector in unit vector notation) 17 students answered correctly to mention some cases. By contrast, the most difficult items were, for example, item 2 (graphic representation of a unit vector), with only 5 students answering correctly, and item 3 (geometric interpretation of dot product) and item 6 (calculation of

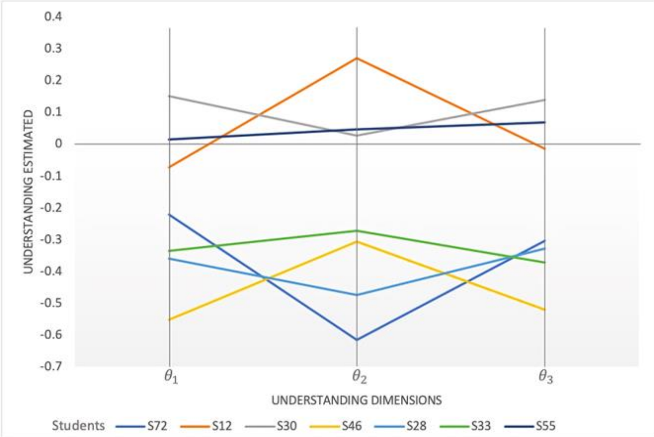

Figure 7. Examples of students with less recurrent profiles (Source: Authors' own elaboration)

dot product using $ABcos\theta$), with only 6 correct answers. An aspect to highlight is that for both students with MMM and HHH profiles, item 2 represented difficulty, and although we did not mention it, only 4 students with LLL profiles answered correctly. This result is similar to that found by Barniol and Zavala (2014a), Rakkapao et al. (2016) and Susac et al. (2018), for whom this item is one of the most difficult on the TUV.

Students with HHH profiles can be supported during instruction; for example, the teacher can organize students into groups and distribute those students who have an HHH profile in each group. Also, the teacher can give individual training to other students and participate in verbal and written ways in front of the class so that the rest of the students listen, observe the procedures the participant uses, and participate in the evaluation process (co-evaluation). These actions enhance the cognitive, social, and cooperative skills of students with HHH profile, and motivate other students to repeat this behavior.

In **Figure 7** we can see the less recurrent profiles, such as MLM (S72), MHM (S12), HMH (S30), and LML (S46). Furthermore, the LLM (S28), MML (S33), and MHH (S55) profiles were only presented once. The behavior in all the mentioned profiles indicates that one, two or the three dimensions of understanding must be strengthened, for example, if the graph of a profile first presents a positive slope and then a negative slope, this means that the student has greater mastery in the dimension of the distinction of a concept. In contrast, if the graph of a profile first presents a negative slope and then a positive slope, it reflects that the dimensions that students dominate are the definition and application of the vector concept. In the case of S55, the profile graph does not present an inflection point, which means the student has a similar understanding of the three dimensions analyzed. In general, the graphic presentation of the profiles can make it easier for teachers and researchers to interpret students' understanding in the three dimensions.

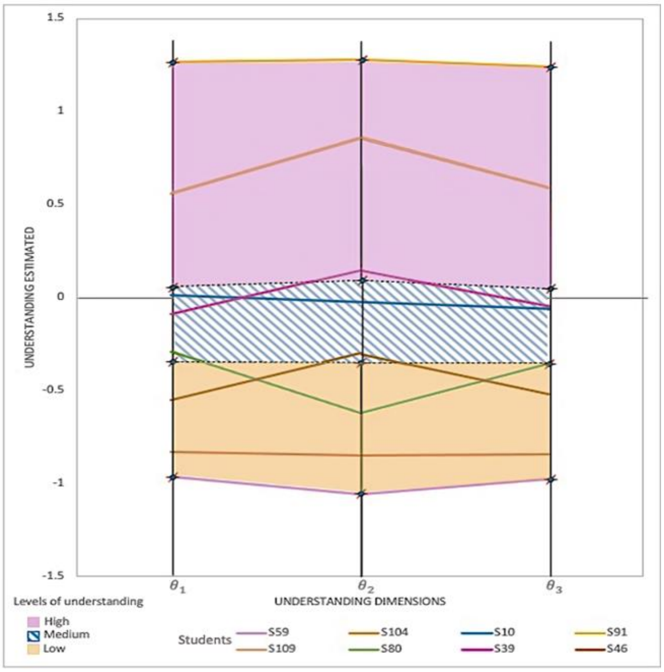

Figure 8. Location of profiles in the understanding levels (Source: Authors' own elaboration)

Finally, in **Figure 8** the levels of understanding are limited by the pink, hatched blue, and yellow areas. Here, we can quickly visualize the students who require more support (for example, S59 and S104) and those who have more abilities and can support other students (for example, S91 and S109). Likewise, students (for example S10) with a medium level of understanding in defining, distinguishing, and applying the vector concept are detected. In cases such as S39, S46 and S80, the dimensions with strengths and weaknesses are identified.

The results obtained in this research, which indicate low, medium, and high levels of understanding of the vector concept among students, suggest the need for a review and adaptation of teaching strategies and curricular development in this area. The diversity in levels of understanding suggests implementing differentiated methodologies that address diverse learning needs. We consider that if a student is classified as having a low level of understanding, then the student has difficulty identifying and applying the basic properties of a vector, as well as performing simple operations such as addition and multiplication by a scalar. A student who reaches a medium level of understanding would be able to handle these operations and recognize the importance of vectors in specific contexts, although they still present difficulties when applying more advanced concepts, such as the decomposition of vectors or understanding their applications in branches such as physics. Finally, a high level of understanding is manifested in the student's ability to apply vector concepts flexibly and creatively in various problems, as well as in his or her ability to connect these concepts with other areas of mathematics and science. These levels indicate that curricular development must incorporate practical and visual activities that facilitate conceptual understanding, as well as the use of educational technology that allows for more interactive learning.

CONCLUSIONS

In this paper, we propose using a Bayesian multidimensional IRT model with three dimensions to estimate the students' understanding of the vector concept. The dimensions considered in the model correspond to the three dimensions of understanding: the operational definition, the distinction, and the application of the concept. Through this model, we can estimate the students' understanding profiles and classify them into low, medium, and high levels.

The results showed that a high percentage of students reached a medium level of understanding, while a low rate of students achieved a high level of understanding. However, there were students with the same level of understanding in the three dimensions and students with different levels in each dimension. It was possible to identify the most complex and straightforward items, of which we observed that it is because the firsts require a more significant number of students' abilities to solve them correctly, compared to the seconds. By this modeling, it was possible to observe those dimensions where students do not reach an appropriate understanding level and those in which they achieve a good understanding level, which can also be noted in the individual understanding profiles of each student.

Likewise, the MIRT model allows us to identify the level of understanding of each student concerning other students; that is, comparisons of understanding between students can be made. The model can provide information about the achievements and weaknesses of students at each level of understanding. According to the results, we provide some strategies to help improve understanding; for example, specific tasks can be designed for each dimension of understanding. To strengthen the dimensions related to the operational definition and application of the concept, it is possible to resort to the use of software such as GeoGebra and Tracker, interactive simulators, or augmented reality systems to model real-life situations with vectors, visualize the characteristics of a vector, and perform vector operations.

To contribute to the dimension of the distinction of the concept from other related concepts, we suggest designing tasks where students identify the differences between scalar and vector, which determine the differences between dot product and cross product, or tasks where students identify the differences between addition and subtraction of vectors, to mention some examples.

Our findings may impact the literature related to understanding the vector concept mainly since, as mentioned at the beginning, this concept has wide applicability in different areas of knowledge. Therefore, knowing each student's understanding profile can be useful for researchers and teachers in designing activities that strengthen each dimension of understanding.

These findings can be applied in real-world educational settings in several impactful ways; for instance, for students who show a low understanding of the operational definition dimension, teachers might implement focused workshops that emphasize practical applications of vectors through interactive tools like GeoGebra, which allow for visual and hands-on experiences that make abstract concepts more concrete. Furthermore, by utilizing the understanding profiles that could help to design the instruction in the classroom, curriculum design, and assessment practices, educators can create more effective learning environments that address the diverse needs of their students.

Finally, in future research, concepts from very different areas of mathematics can be addressed, such as algebra, differential equations, statistics, spatial geometry, and differential and integral calculus, to mention a few. These studies could provide valuable information about the mathematical skills of different populations, as well as the effectiveness of different teaching methodologies. In addition, different IRT models can be explored to evaluate the effectiveness of the model.

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