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Mathematics teachers' professional experimentation with mathematical origami in secondary education

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Abstract

Mathematical origami can be a valuable tool for teaching mathematics. This study explores mathematical origami teaching practices in secondary education. The addressed research question is: *What do mathematics teachers report about their professional experimentation with mathematical origami in the secondary education classroom?* The study focuses on exploring and describing these professional experiments. Using grounded research theory, we analyzed 45 articles written by teachers from four countries. Four key categories emerged from our analysis: *mathematical topics taught using mathematical folding, teachers' reasons for using folding, aspects of a teaching process*, and *advice for teachers*. In a validation process, experts confirmed these categories. Our findings indicate that mathematical origami is applied across a broad range of topics and domains, with various reasons for its use, and diverse learning objectives achieved in different lesson phases. Teachers also encourage their colleagues to incorporate folding into their classrooms and provide valuable advice for implementation.

Keywords: mathematical origami, grounded research, professional experimentation, mathematical folding, secondary education, professional publications

INTRODUCTION

Numerous books suggest that a wide range of mathematical subjects can be explored through mathematical origami, as demonstrated in the centennial work of Row (1901) and Friedman's (2018) comprehensive historical overview. In our research, we are interested in how the Asian tradition of origami is implemented European mathematics classrooms. While most origami-based teaching focuses on geometry, recent studies show that origami has made contributions to fields such as calculus (Hull, 2013), logic (Serre & Spreafico, 2018), fractals (Bahmani et al., 2014), and axiomatizing (Nedrenco, 2018). In recent years, an increasing body of research points to mathematical origami as an inspiring resource for learning mathematics. Beyond the various mathematical subjects covered in the literature, evidence suggests that folding activities help improve students' spatial visualization skills (Boakes, 2009) and spatial thinking (Burte et al., 2017). Moreover, it has been shown that folding can positively influence students' attitudes toward mathematics (Serre & Spreafico, 2018) and provide a positive learning experience (Boakes, 2009). It can also address the needs of diverse student groups, such as visually impaired children (Moratelli Pinho et al., 2016), children with special needs (Vardi & Golan, 2009), or students with math anxiety (Meyer, 2020). The use of origami to learn a mathematical topic can be combined with technology (Klemer & Rapoport, 2020), or folding is sometimes only a small part of the total series of mathematics tasks (Kafetzopoulos & Psycharis, 2022). However, our impression is that mathematical origami is not widely integrated into everyday classroom practices in secondary education. Though research on the number of teachers that use mathematical origami is not available, research shows that mathematical manipulative materials are hardly used by teachers in

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Contribution to the literature

- This study draws on articles written by teachers, providing an overview of their approaches, insights, and findings with mathematical origami.
- This study contributes to the knowledge about applications for using mathematical origami in secondary education.
- This study contributes to the knowledge about reasons for using mathematical origami in secondary education.

secondary education compared to primary education (O'Meara et al., 2020). So, although it is indicated that mathematical origami is promising for learning mathematics, it is rarely applied. This study explores whether and how we can connect mathematical origami to teachers' teaching practices in secondary education and inform our future studies. To do so, we aim to study teachers' professional experimentation. This is part of a multi-year educational design research project investigating how teaching with mathematical origami, with its adaptability to a range of learning objectives, contributes to mathematics learning in secondary education.

For this study, our interest is in those mathematics teachers who fold in their secondary education lessons. By folding during their daily practice, these teachers are experimenting professionally. Some teachers share their classroom folding experiences and insights in articles in teacher journals. To research mathematical folding in the classroom, we focus on professional publications about teachers' folding experiments.

To study teachers' professional experimentation with mathematical origami in secondary education, we use grounded theory, analyzing professional publications of teachers in secondary education. For a sample of suitable size, we extended our research to professional publications in four countries: France (FR), Germany (GE), the United Kingdom (UK), and the Netherlands (NL). We use professional knowledge to inform the development of academic knowledge. To improve the validity of our results, we discussed the developed codebook with expert teachers of mathematical origami. This research has led to four interesting categories that give a comprehensive view of teaching with mathematical origami in secondary education.

In the next section, we will first consider the choice of professional publications and elaborate on mathematical origami.

THEORETICAL BACKGROUND

Professional Publications

To clarify the approach of using professional publications, let us discuss the model for teacher professional growth–with a slightly adjusted layout–of Clarke and Hollingsworth (2002, p. 957), depicted in **Figure 1**. The prime point is that we want to study how

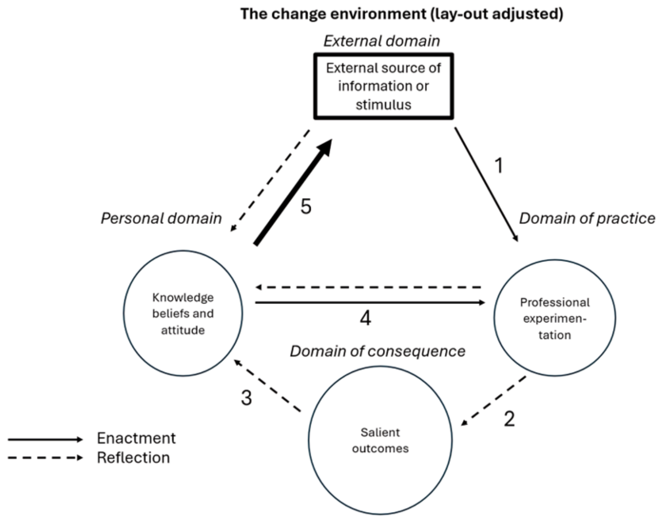

Figure 1. Professional growth from professional experimentation (Clarke & Hollingsworth, 2002)

teachers' knowledge, beliefs, and attitudes about folding (personal domain) are reflected in their journal papers: the external sources they create for their peers (external domain).

Clarke and Hollingsworth's (2002) model describes professional growth. Practicing mathematics teachers continue to develop and improve their teaching over time (Goldsmith et al., 2014). According to Clarke and Hollingsworth's (2002) model, teachers can be motivated by external sources of information (external domain) to engage in professional experimentation in the classroom, in our case, with mathematical origami (arrow 1). Reflection on the salient outcomes in **Figure 1** (arrow 2) leads to a reflection of teachers' personal beliefs (arrow 3) and more experiments (arrow 4). For some teachers, the result is that they spend time and effort writing about it in professional journals for teachers (external domain), enacting on the external domain, arrow 5 in the model. Providing this external source of information via an article in a mathematics teacher's magazine might influence other teachers to start with professional experimentation (back to arrow 1). What teachers publish about their professional experimentation in the external domain can catalyze professional growth and be a significant source of professional knowledge on folding.

Mathematical Origami

To organize and select samples in our study, we propose four subdivisions of mathematical origami.

Traditionally, there were about a hundred origami designs, some of which allowed for modifications such as cutting the paper to add an extra leg to a frog or other animal. Since the 1980s, computer programs have been developed to design origami figures, leading to a dramatic increase in the number of origami models. With this explosion of designs, the rules of 'classic' origami have also become much stricter: no cutting, no gluing, and the model must be made from a single sheet of paper. These stricter rules, along with increasingly complex crease patterns, have introduced more and more mathematics into origami, including conjectures that can be proven. Some educators have developed and implemented an entire series of lectures on mathematical origami (Boakes, 2009; Demaine & O'Rourke, 2007; Golan, 2011; Hull, 2013; Meyer & Mukoda, 2021; Tubis & Mills, 2006). Origami-based task designs for mathematics exist on various educational levels, from elementary school (Burte et al., 2017; Golan & Jackson, 2009) to university (Demaine & O'Rourke, 2007). The connection between origami and mathematics is also reflected in the names of the books by Haga (2008), "Orgamics" and "Origametry" by Hull (2021), both cited in Nedrenco's (2022) historical classification. As Nedrenco's (2022) research concerns learning axioms via the process of paper folding, he does not include mathematical figures that are first folded and then observed in his classification. As we are more broadly interested in teaching various mathematics topics supported by paper folding in secondary education, we propose a classification based on *teaching* mathematics supported by origami.

We suggest the following four categories (see **Figure 2**):

(1) teaching mathematics based on the design of origami models,

(2) teaching mathematics while folding origami figures,

(3) teaching mathematics while folding following specified rules, and

(4) teaching mathematics after folding, by handling an origami model.

The distinction between these four subdivisions is essential because each represents a different teaching approach, and another use of the folding process results. For example, inquiry-based learning is possible *during* folding in subdivision 2 and subdivision 3 and *after* folding in subdivision 4. Next, we elaborate on these four subdivisions.

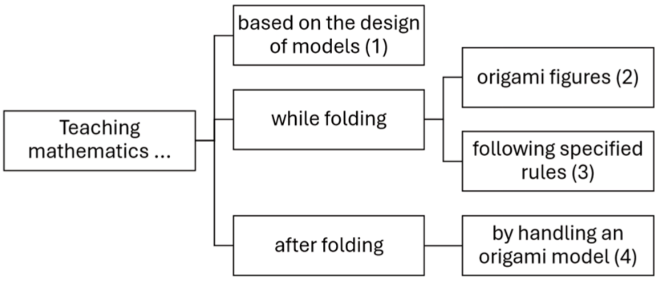

Figure 2. Teaching mathematics with origami (Source: Authors' own elaboration)

Teaching mathematics based on the design of origami models

This viewpoint is taught and described, amongst others, by Demaine and O'Rourke (2007), Hull (2021), and O'Rourke (2011). When developing computer software to design origami figures, several agreements needed to be made, for example, the paper would be folded in the right direction and not intersect with itself. These rules are now translated into origami axioms, and Lang (2010) has proved their completeness. Within this axiomatics approach, exciting properties and propositions can be proved, like 2-colorability, flat foldability, the difference between the number of mountain- and valley folds at an intersection of crease points, and many more mathematical proofs on origami folding; see for examples Demaine (2024).

Teaching mathematics while folding origami figures

This viewpoint is taught amongst others by Golan and Jackson (2009), Meyer and Mukoda (2021), Boakes (2008), and Tubis and Mills (2006). Golan (2011) teaches mathematics to children between 4 and 12 years old by letting them fold origami figures. Meanwhile, the correct mathematical language describes and questions the folds, shapes, and symmetries that appear. Next, students can be asked to calculate the angles and surfaces of the figure at hand. The goal here is for students to fold a nice figure while at the same time playfully encountering mathematics. Meyer (2020) uses the same approach for college students. The mathematical content is adapted to the age of the students, for example, calculations with square roots and properties are covered. This approach is also followed by Boakes (2008). The books for mathematics teachers by Tubis and Mills (2006) deal with origami topics, such as variations in origami boxes and their mathematical aspects.

Teaching mathematics via mathematical paper folding following specified rules

This viewpoint is based on the definition of Nedrenco (2022), first published in 2018. He describes the re-invention of origami axioms. He provides a definition of mathematical folding to distinguish between the process of folding and the use of paper manipulatives (see subdivision 4). In his definition (p. 258) he writes about "declared ... describable rules", which we think is too restrictive for secondary education. We use the words "following specified rules," which results in the following definition: *By mathematical paper folding, we understand a branch of paper folding, where paper is being folded following specified rules, with the goal to analyze the folding process and result mathematically.*

In this way, one can include a folding task of Fröbel, where students are given a piece of paper with an organic shape and the inquiry-based task "fold a square", based on the first task in Row (1901). When the students have completed folding, a teacher can ask: "How do you know it's a square?", thus leading to mathematical reasoning about the properties of a square. Many tasks of Row (1901), Hull (2013) and Haga (2008) can be described as this form of mathematical folding. To make a simple distinction with subdivision 2: the aim of subdivision 2 is to take a nicely folded figure home after the lesson, and after subdivision 3, most of the folded figures will end up in the paper bin, as the focus is on the process and not on the result.

Teaching mathematics by handling an origami model

Nedrenco (2018) gives an example of folding and gluing a pyramid, after which the volume and area of the pyramid are calculated. So, in this view, a clear distinction exists between making the pyramid and doing mathematics. The same goes for the way Petzschler and Etzhold (2014) use paper folding to collect data for research on combinatorics and probabilities by students. This folding type aims for students to construct a manipulative, later used to do actions like calculations, gather data and obtain more insight into mathematical (3D) figures.

Of course, the real world of mathematical origami is more challenging to classify. An example of this is the hexaflexagon (**Figure 3**), discovered and described by the mathematician Arthur Stone (subdivision 1), as mentioned by Mitchell (2024). It can be used to teach students about 60-degree angles and isosceles triangles and still be nice to take home (subdivision 2). The folding process can be used to prove the similarity of the triangles (subdivision 3), and handling the hexaflexagon after the folding process can serve as a basis for statistics (subdivision 4). So, this grouping into four subdivisions is driven by folding in teaching.

Using this theoretical background in our study, we want to connect with teachers' professional experimentation in mathematical origami to map out professional knowledge in this area as a starting point for academic knowledge. The research question in this paper is: *What do mathematics teachers report about their professional experimentation with mathematical origami in the secondary education classroom?*

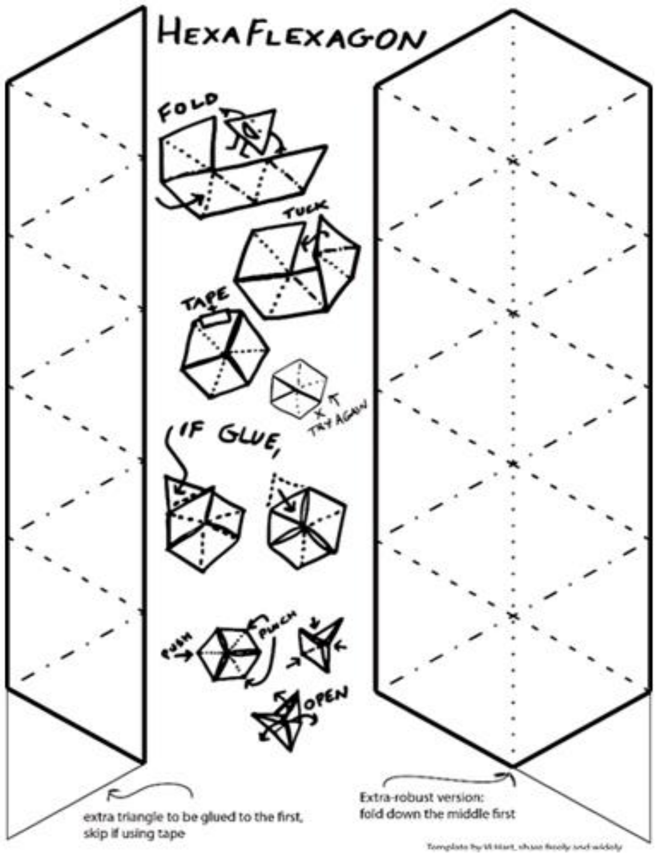

Figure 3. Making a hexaflexagon (Hart, 2024)

To conduct this study, we used a grounded research approach to investigate what teachers write about professional experimentation with mathematical folding in their classrooms in the domain of practice, the salient outcomes in the domain of consequences, and the lessons learned in the personal domain. To answer the research question, we analyzed articles written by teachers from FR, GE, the NL, and the UK.

METHOD

Research Design

The qualitative approach we used for our data collection and analysis is the grounded theory approach, described by Boeije (2009). As this is the first study we know of researching this subject of secondary education teacher-written articles on mathematical origami, the focus is exploring and describing the field of study. The constant comparison of data with the emerging categories gives a more abstract and conceptual model that can be grounded in the data. We applied three rounds of coding: open, axial, and selective (Boeije, 2009). Next, to verify the findings of this grounded theory, we used a form of member validation by presenting the findings during interviews with teachers with experience in mathematical folding, referred to as 'experts' from now on.

Sampling, Data Collection, and Data Preparation of Articles

To make our sample more extensive and to obtain a realistic impression of mathematical folding in mathematics education beyond our own country, we included contributions from French, German, Dutch, and UK mathematics teachers. So, articles from these countries are selection criterium 1. All articles should be written for teachers in secondary education (selection criterium 2). As we assume that teachers in secondary education want to share lesson ideas with their fellow teachers in a convenient way, the articles had to be written in the country's language (selection criterium 3). The analysis of teacher-written articles has the disadvantage that the articles cannot be accessed through academic databases, so the articles had to be collected per country, usually through a teacher association. We now list a summary of the inclusion criteria of the associations of mathematics teachers and journals: For the French journals, we started with the French Teachers Association APMEP and its journals: PLOT (until 2017), Bulletin Vert (until 2017), and Au fil des Maths (from 2018). Snowballing from these contributions, we also included some commercial journals for mathematics teachers aimed at secondary education from Tangente Éducation, based on the summary on the website. Search words were *plier* and *origami*. We used the websites www.apmep.fr and www.tangente-mag.com. We found 114 articles on the APMEP site, 8 from Plot/Bulletin vert/Au fil des math, and 8 from Tangete, 130 in total. After filtering out 58 articles with words like "multiplier" (multiply), not on paper folding and/or doubles, we uploaded 72 articles in software for analyzing qualitative research, MAXQDA, for quick reading.

For the German journals, we started with the German Teachers Association MNU, for which we took a membership, giving access to a library of previous and present *MNU-journals* of MNU. From snowballing these contributions, we also included some commercial journals for mathematics teachers aimed at secondary education. A subscription to the commercial journal *Mathematik Lehren*, gave access to a library of previous and present journals. We accessed some articles from the journal PM: Praxis der Mathematik in der Schule (until 2017) through an inter-library connection. Search words were *papierfalten, falten,* and *origami*. We used the websites http://www.mnu.de and www.friedrich-verlag.de/mathematik/mathematik-lehren. We found 132 articles on the MNU site and journal, 10 in Praxis der Mathematik, and 38 in Mathematik Lehren, 180 in total. After filtering out 125 articles with words like "entfalten" (develop), not on paper folding and/or doubles, we uploaded 55 articles in MAXQDA for the first quick read.

For the UK journals, we started with the UK Teachers Association ATM and their journal MT: *Mathematics Teaching*. Next, we used Utrecht University's access to the JSTOR archive to access the MiS: *Mathematics in School* journal for secondary education of the Mathematical Association. Search words were *origami* and *folding*. We used the websites https://atm.org.uk/ and www.m-a.org.uk/mathematics-in-school. We found 17 articles on the UK Teachers Association site and 24 in the Mathematics in School journal, 41 in total. These articles were uploaded in MAXQDA for the first quick read.

We started with the Dutch Mathematics Teachers Association NVvW and its journal Euclides for the Dutch journals. As members of this society, we had access to present and previous journals. Search words were *origami* and *vouw*. We used the website: www.nvvw.nl/euclides/. We found 126 articles in the complete archive on the Euclides site. After filtering out 41 doubles and 8 articles not about folding, we uploaded 78 articles in MAXQDA for the first quick read.

The articles were selected between 2022 (FR) and 2023 (the NL). In **Figure 4** we present a flowchart of the selection process after a quick read in MAXQDA with the number of articles.

Articles that only contained a straightforward folding task, without explanation or additions about the use in the classroom, or articles about the use of folding in a tinkering task were removed from the sample. So, selection criterium 4 emerged: the article had to include a description of a mathematical folding task, combined with some educational comments. Some articles in our sample were not written for secondary education (selection criterium 2). The student group was not always clearly indicated in the article, and class numbering of primary and secondary education is inconsistent throughout Europe. We also found that some articles were solely written by teacher educators or other professionals who were not personally involved in teaching mathematics in class at secondary education (selection criterium 1). We tried to determine whether the authors were secondary education teachers by conducting searches online. For example, if an article was written by a teacher in collaboration with a teacher educator, we included it in our sample. A list of the references for the resulting 45 articles is provided.

To code the text of the articles in MAXQDA, they had to be in PDF. Some sources consisted of scanned images, so the articles had to be transformed into Word and PDF. This transformation resulted in a (sometimes major) loss of accuracy of the text of the transformed articles, so the selected samples had to be re-checked and adjusted manually using the original images.

In summary, in our research on professional experimentation with mathematical folding, we investigated teachers' knowledge, beliefs, and attitudes about mathematical folding (personal domain). We did this via articles in teachers' magazines where teachers

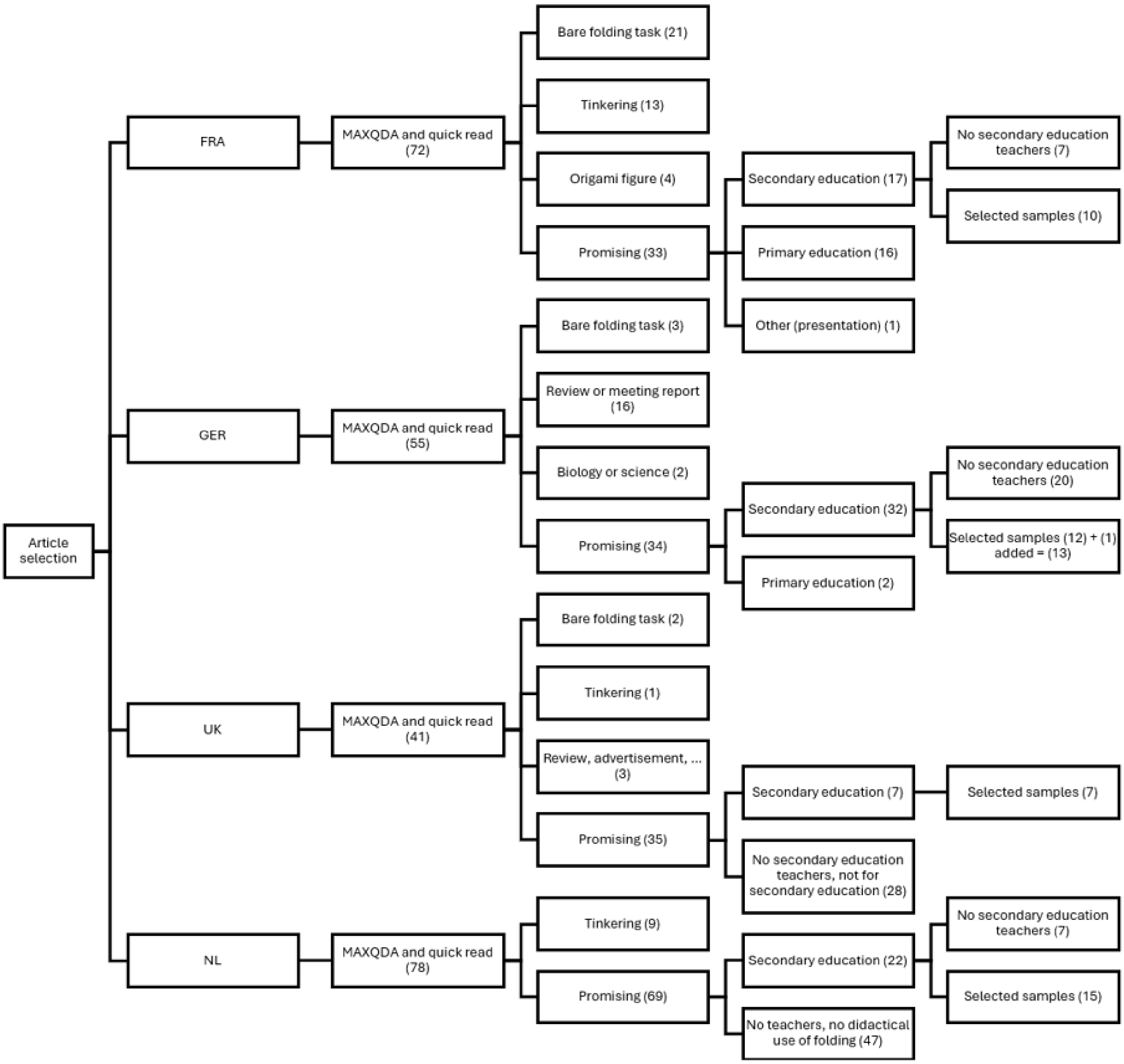

Figure 4. Article selection coding process (Source: Authors' own elaboration)

deliberate on mathematical folding tasks for their colleagues (external domain). Therefore, we had four selection criteria: the articles should be written by French, German, Dutch, and UK mathematics teachers, for teachers in secondary education, in the country's language and describing a mathematical folding task, combined with some educational comments.

Sampling Interviewees, Data Collection, and Data Preparation Interviews

To check the validity of the framework, we used interviews with experts. We selected them from a similar cohort of teachers as the writers of the professional publications. Though the writers and interviewed experts are not the same, by selecting them from the same cohort, we have a form of "member validation" of our coding (Boeije, 2009, p. 177).

We searched within and outside our network for experts on mathematical origami, as much as possible, within category 3 of mathematical folding. We invited six experts within and outside the European Union; five agreed to an interview. Ethical approval was obtained from the Science-Geosciences Ethics Review Board from the Utrecht University for the questionnaire and the number of participants. **Table 1** describes the main expertise of the interviewees.

The interviews were semi-structured. The experts were first asked open questions about their experience with teaching in general and with mathematical folding when teaching. Next, we asked them about their reasons for using mathematical folding, in what phases of the learning process they used it, and the pros and cons they experienced with it in the classroom. Finally, their opinion was asked on the categories and subcategories that resulted from the coding of the articles. One interview was conducted live and recorded with a voice recorder. The other four interviews were recorded online via Teams or Google Meet. All interviews were transcribed using Amberscript and then improved manually.

Data Analysis

We first performed the open and axial coding steps on the French articles. Two temporary codes quickly emerged. One was inspired by the personal domain: "why," which contained reasons to use mathematical folding. The other was inspired by professional experimentation: "what," which contained text excerpts of mathematical topics that were covered with folding. The second author evaluated one of the emerging categories, and agreements were obtained on the subcategories. Subsequentially, the coding method was better defined as:

(1) the unit of analysis should be one or more complete sentences (capital letter to full stop) per code and

(2) more codes may be assigned to a unit of analysis.

During a discussion on the coding process with the team during the axial coding phase, we reassigned categories and agreed on four major categories and a first classification of subcategories. In the first round of the selective coding phase, one full article was coded by the first and second authors. After discussing the differences in coding, full agreement was reached. Next, we discussed the various categories with all four authors and agreed on a first version of the categories' classification, names, and descriptions. When we collectively agreed on the categorization, we ran the different coding cycle steps for the German articles. These first two countries served as material for the grounded theory approach to arrive at a final coding scheme for the articles from FR, GE, the UK, and the NL.

The final coding rounds with the final coding scheme were done with 45 articles from the four countries. In comparing the assignment of codes across articles from different countries (i.e., looking at a series of assigned codes to one subcategory) by the first and second authors, several excerpts were switched to another subcategory. The names of the categories were adjusted for conformity.

The transcriptions of the interviews with experts were coded in MAXQDA using the codebook of the teachers' articles.

RESULTS

In total, using a grounded research approach, we have coded 513 excerpts, which are distributed over 45 articles from four countries: FR, GE, the UK, and the NL. We aimed to explore teachers' articles about professional experimentation in the domain of practice, the domain of consequences, and the lessons learned in the personal domain with mathematical folding in their classrooms. We have listed the categories that emerged from the coding process according to the number of excerpts we have coded in the four countries and summarized them in four tables. The first category that emerged in our analysis contains the most excerpts: *mathematical topics taught using mathematical folding*. The second category is teachers' *reasons for using folding*, category 3 is *aspects of a teaching process*, and category 4, *advice for teachers*. After each of the tables, we provide examples of excerpts characteristic for subcategories in the table. The examples we have chosen are, to the furthest extent, excerpts that can be read autonomously, so, next to illustrating the subcategory, they also sketch a picture of the mathematical task that underlies them. These examples also contain cases where multiple categories are assigned to the same excerpt.

Tables and Excerpts of Mathematical Folding

Mathematical topics taught using mathematical folding

The first category, with the most excerpts that emerged, concerns the mathematical topics taught using mathematical origami. This category shapes the overall picture of what teachers report about mathematical origami. It aligns with their goal of writing articles for fellow educators, aiming to inform them about potential topics for using mathematical origami in the mathematics classroom. It also connects to the "knowledge" aspect within the personal domain, as the authors demonstrate their expertise in mathematical origami applied to specific mathematical topics. Additionally, this category provides insight into their professional experimentation within the domain of practice. To maintain clarity in the presentation of subcategories, we set a minimum threshold of seven excerpts for a subcategory to be included in the table. This criterion resulted in four to six subcategories per category.

As stated in the introduction, the tasks prevalently involve *geometry*. It is the largest category, with almost half of the number of excerpts, followed by *relationships and functions* (**Table 2**).

For example, in an article of Pietsch (2007), several (sub) categories are covered, like *geometry (1.1); triangle,* and the category *proof (1.4)*, showing the students that the sum of angles in a triangle is 180 degrees (**Figure 5**).

Pietsch (2007) writes that the three internal angles appear to join together seamlessly at the base of the height to form a straight angle of 180° (p. 13). This example is a kind of proof without words: with three folds, the angle sum is made clear. The "point-to-point" and "line-to-line" folds (see **Figure 6**), also found in

Table 2. Mathematical topics taught using mathematical folding

Label and description	FR	GE	NL	UK	**Total**
1.1 Geometry					
• Triangles	8	5	1	8	**22**
• Angles (bisector, right angle, …)	8	5	2	3	**18**
• Lines	10	3	5	0	**18**
• Spatial figures	9	0	5	2	**16**
• Polygons (quadrilateral, pentagon, …)	2	2	1	11	**16**
• Other topics	5	1	2	4	**12**
Totals 1.1	42	16	16	28	**102**
1.2 Relationships and functions					
• Calculating and algebra	5	0	3	6	**14**
• Optimization	2	2	4	0	**8**
• Exponential	1	2	2	2	**7**
• Other subtopics	12	1	4	4	**21**
Totals 1.2	20	5	13	12	**50**
1.3 Mathematical topics that arise from folding (origami)					
• Huzita/Hatori/Justin axioms	10	0	0	0	**10**
• Other topics like to solve problems beyond ruler and compass …	17	0	1	1	**19**
Totals 1.3	27	0	1	1	**29**
1.4 Proof. Folding tasks used as a (general principle requiring a) proof	1	8	2	10	**21**
1.5 Other mathematical topics. Axiomatizing, approximation, vectors, making curves, graphs, probability, and statistics	8	1	6	2	**17**
Totals overall	**98**	**30**	**38**	**53**	**219**

Note. Mathematics topics that can be explained or explored using mathematical folding. The depth of elaboration in the excerpt ranges from mentioning the mathematics topic to describing the activity in a mathematics lesson

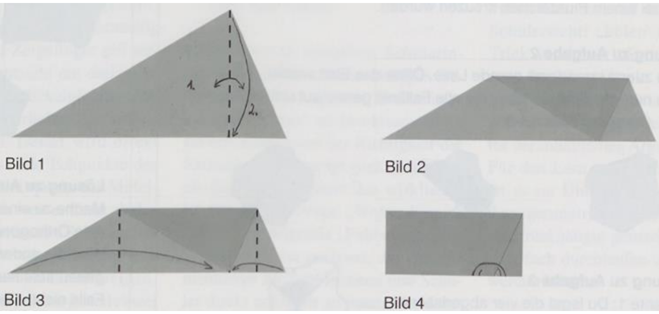

Figure 5. Folding the sum of angles of a triangle (Pietsch, 2007, p. 16)

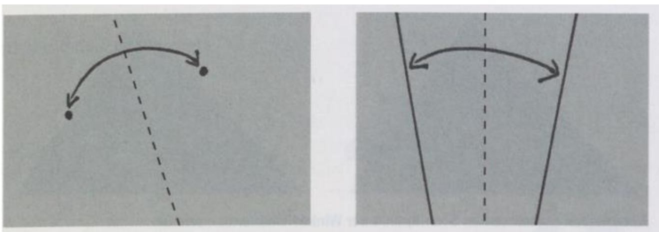

Figure 6. Folding point to point and line to line (Pietsch, 2007, p. 13)

Pietsch's (2007) article, correspond geometrically to the construction of perpendiculars and bisectors because, in both cases, the fold line corresponds to the axis of a reflection. The construction of these lines by one fold is to be done faster than the construction by pencil. Point-to-point and line-to-line are both also *mathematical topics that arise from origami (1.3)*: axioms 1 and 3 (Lafond, 2013). These axioms follow from the (im)possibilities of folding

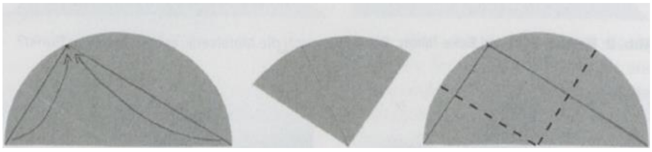

Figure 7. Folding the Thales theorem (Pietsch, 2007, p. 16)

origami, and teaching mathematics using the axioms fits into our category of section 2.2 of this paper: "*Teaching mathematics based on the design of origami models*".

Proof (1.4) is a part of mathematics that students in secondary education consider quite difficult. About proving the Thales theorem with folds (**Figure 7**), Pietsch (2007) writes that

> "… this reasoning is no less mathematically precise than the usual proof using two isosceles triangles. When folding, however, the usual (from the student's point of view: "nasty") heuristic trick of suddenly bringing the radius to the peripheral point into play, can be dispensed with. Instead, the actions themselves prepare the structure of the substantive argumentation" (p. 14).

This excerpt is interesting because the teacher indicates that simple folds can contribute to the understanding of a proof, and that making the folds contributes to the argumentation of the proof, thus interconnecting very tangible actions with sophisticated mathematics.

For another important part of the secondary education curriculum, *relationships and functions (1.2),* multiple folding tasks are described. For the subcategory *optimization*, we present an example of a teacher who uses folding to have the students find out the most extensive volume possible from a sheet of paper and create a formula for it:

> "In this task, four (square) corners must be cut from a rectangular sheet of paper. This becomes the construction sheet of a box without a lid. Then, the content must be calculated. [Students are] asked: the dimensions of the box with the largest content. This is a wonderful moment. I hand out to all students a (colored) A4 sheet and have them fold a box, without a lid […] You get a whole bunch of boxes. Then, you immediately see that the low edges give little content, but the high edges do as well. It also becomes clear within which dimensions you can still get a real box […] The content is $x\,(L-2x)(W-2x)$. Next steps are expanding the formula and differentiating, using the ABC formula. Eventually, the best answer will come" (van Oord, 2004).

This excerpt shows that folding can make complex secondary education mathematics directly observable. The teacher has all students fold different boxes, to show that an A4 sheet can be used to make numerous boxes with different volumes. To calculate if the largest volume exists, the students set up a function and determine the maximum volume.

A last example is learning about vectors, a subject typically found in *other mathematical topics (1.5)*. By using a 3D folded coordinate system with wooden sticks and strings, students actively engage with the concept of vectors (see **Figure 8**). Teachers believe this hands-on approach helps students experience vectors and their applications in a dynamic, action-oriented way (Casteli & Trahe, 2016).

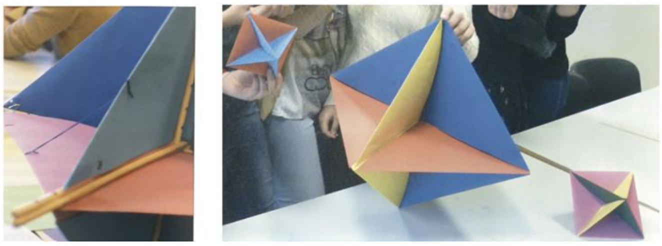

Figure 8. Paper manipulative to learn vectors (Casteli & Trahe, 2016, p. 36)

Teachers' reasons for using folding

The second category that emerged is "teachers' reasons for using folding" (**Table 3**). This category emerged rather quickly after category 1, as most articles describe which mathematical topics are covered (**Table 2**) and then elaborate on why the approach via mathematical origami was chosen. These explanations have led to this category about the reasons for using folding. These reasons provide–amongst other things–an insight into the beliefs and attitudes of teachers in the personal domain via the salient outcomes of the described professional experimentation.

Some reasons for applying mathematical origami in class have already been mentioned in the excerpts on the mathematical topics from **Table 2**. For example, in the excerpt of Pietsch (2007) about the Thales theorem, the actions themselves prepare the structure of substantive argumentation. This is a reference to category *(2.1), to actively explore mathematics manually* because the teacher makes a noticeably clear connection between the action of folding and the structure of the argumentation that follows from it. An example of *(2.2), to visualize mathematics* is, as follows: "… this is followed by the perpendicular position, of which the meaning is only captured visually by folding a paper twice in half" (Vredenduin, 1963). So, by folding a piece of paper twice, an angle of 90 degrees is demonstrated, leading–in several symbolic steps, see **Figure 9**–to proving that if three corners of a square are perpendicular, then so is the fourth.

Table 3. Teachers' reasons for using folding

Label and description	FR	GE	NL	UK	**Total**
2.1. To actively explore mathematics manually. Invite students to manipulate paper by hand and/or mathematically interpret the action of folding.	12	9	5	1	**27**
2.2 To visualize mathematics. Invite students to interpret the visual aspect of folding mathematically	9	12	3	3	**27**
2.3 To ground understanding. Invite students to make sense of a mathematical concept by connecting it to folding.	6	15	2	4	**27**
2.4 To approach mathematics in a non-standard way. Invite students to approach mathematics in a non-standard way	7	10	4	6	**27**
2.5 To create positive effect. Invite students to engage, improve motivation, develop a positive attitude to mathematics, and show commitment.	7	7	4	3	**21**
2.6 To implement low floor/high ceiling tasks. Invite students to experience the possible profundity of mathematical topics building on simple folding tasks.	7	7	0	2	**16**
Totals overall	**48**	**60**	**20**	**19**	**145**

$$A' \,//\, A \perp B \,//\, B' \Rightarrow A' \perp B',$$
$$A \perp B \,//\, B' \Rightarrow A \perp B',$$
$$A \perp B \perp C \Rightarrow A \,//\, C,$$
$$A \perp B \perp C \perp D \Rightarrow A \perp D \; ($$

Figure 9. On proving the characteristics of a square (Vredenduin, 1963, p. 239)

To ground understanding (2.3) and *positive affect (2.5)* are both illustrated by a French teacher via the next excerpt:

> "Above all, manipulating this object helps to grasp certain properties of the regular tetrahedron better: observing the orthogonality of opposite edges, before demonstrating it, enchants students and helps to maintain their enthusiasm for any other question relating to this solid" (Aoustin, 2021).

This teacher describes that manipulating objects helps to grasp properties better and thus establishes a clear link between handling an object and the understanding of the object. In addition, the teacher indicates that using the paper object also enthuses students for a longer period. There are more examples on *grounding understanding* (2.3), where some teachers are quite clear in their statements, like a Dutch teacher:

> "Did you ever have the experience that students suddenly "grasped" the material much better after you had them cut, paste, or fold during a lesson? I had this experience most strongly when […] I had folded the parabola as a set of tangents in my math D class" (Boels, 2016).

Another example is English teachers who state that "students gained interest [via a folding task], and it made them think on a deeper level. Concepts like faces, edges, vertices, and other mathematical terms were grasped and understood" (Leroux & Santos, 2009).

Both excerpts concern making mathematics something 'you can take hold of' by folding paper.

In an excerpt about folding boxes at **Table 2**, van Oord (2004) indicates that this is a 'wonderful moment'. van Oord (2004) points at the introduction of the colored sheets, with which the boxes will be folded, which means for the students that *mathematics will be approached in a non-standard way (2.4)*, but also that a positive atmosphere is created in the classroom, so *(2.5), create positive affect.*

In several previous excerpts, it has already been shown that a few simple folds can lead to complex mathematics (Boels, 2016; Pietsch, 2007; van Oord, 2004). The possibility of starting with a simple folding task, and, depending on the student's interest, ending up at a different level of difficulty for the students, is also described by teachers in *to implement low floor/high ceiling tasks (2.6)*. The following excerpt gives an example of a task where students start folding and calculating bicimals, which can be expanded to a discussion about rational and real numbers.

> "Discussion can involve what happens if the point falls on a fold, and whether points must lie on some fold sometime. Does each bicimal define a unique point? Conversely, does each point correspond to some unique bicimal? In the background to this is the idea of recurring and non-recurring bicimals, and so the rational and real numbers. […] Construct and investigate other non-recurring and recurring forms" (Brissenden, 1972).

Aspects of a teaching process

In addition to category 2 with reasons for mathematical origami, which mainly focuses on the learning of the students, a group of excerpts was also created that further shape the professional experimentation in the domain of practice by indicating where within the teaching process, mathematical origami can be given a place. This third category is "aspects of a teaching process" (**Table 4**). This category helps us to interpret professional experimentation with mathematical origami because the teaching processes using mathematical origami become sharper defined in this way.

The *transition from informal to formal (3.1)* is a part of the teaching process in which students are guided from the concrete, tangible, and informal way of doing mathematics to a more abstract and formal way of doing mathematics with increasing preciseness. This can also involve the use of more appropriate mathematical language, and the introduction of mathematical symbols, as illustrated in the following excerpt:

Table 4. Aspects of a teaching process

Label and description	FR	GE	NL	UK	**Total**
3.1 To transition from informal to formal. Allowing students to use folding as an intermediary between informal and formal and develop mathematical language.	9	28	5	1	**43**
3.2 To practice skills. Allowing students to practice mathematical and 21st century skills based on the folding task.	9	15	4	6	**34**
3.3 To elicit prior knowledge. Allowing students to re-activate prior knowledge through a folding task.	4	8	1	0	**13**
3.4 To encourage reflection. Allowing students to reflect on the mathematical topic at hand.	4	4	4	2	**14**
Totals overall	**26**	**55**	**14**	**9**	**104**

Table 5. Advice for teachers

Label and description	FR	GE	NL	UK	**Total**
4.1 On implementing the task. Practical tips and specifics while teaching the lesson.	12	5	1	0	**18**
4.2 On reflecting on the implemented task. Retrospective of the lesson and/or improvements for the next time.	3	4	5	0	**12**
4.3 On preparing the folding task. Matters that should be covered before class.	2	3	1	2	**8**
4.4 On encouraging and preparing colleagues to use folding in classroom. Practical tips and specifics to involve other teachers.	0	0	2	5	**7**
Totals overall	17	12	9	7	**45**

Note. Advice from the authors on implementing the folding activity in the classroom

> "The students notice that the folding is not very precise. They take a blank square sheet of paper and trace the folds with a ruler. This is an opportunity to use all the geometry vocabulary: you must draw segments, place their midpoints, and draw parallels and perpendiculars. There are right angles and other angles. And to get the whole class talking about the same point–brilliant invention–we give the points names!" (Barthelet, 2021).

First, the fact that students feel that the folding itself is not yet precise enough, and that they want to make it more precise, is a step towards more formal mathematics. Subsequently, it turns out that the students, by discussing the folding task, feel the need to introduce names–and thus a possible formal notation: capital letters–at the various folding points. The subcategory *practice skills (3.2)* is characterized by 21st century skills like reflection, cooperation, and discussion, and skills that are associated with mathematics like figuring out, constructing, working neatly, and solving problems. In the following example, it is indicated that the task of folding an origami cube eventually works towards mathematical argumentation: "This article aims to promote the competence of mathematical argumentation, which is illustrated by a propaedeutic development of the concept of congruence in grade 6, using the example of the origami cube" (Kaufmann, 2016, p. 12).

The subcategory *to elicit prior knowledge (3.3)* is usually not mentioned as the primary purpose of folding, but often as a "by-catch" in a geometric folding task, with bisectors and other special lines and figures. This subcategory also occurs in the topic of graphs, shown in this excerpt: "Next, you can then show how you make a Hamilton circular walk in the flat graph of the cube. This is also a great opportunity to brush up on knowledge about graphs" (Tap, 2014, p. 14). So, brushing up on the knowledge of graphs is a by-catch of the Hamilton circular walk, based on the folding task of making Hulls' (2013) bucky ball. The last category, *to encourage reflection (3.4),* can be illustrated by the questions in the next example of an excerpt about folding a pentagon: "But does this give you the perfection of the Pythagoreans? Is the pentagon regular? These questions are the starting point for argumentative considerations that can be carried out "dynamically" in various ways" (Etzhold & Petzschler, 2016, p. 30). This reflection is related to the practice of 21st century skills, such as learning how to build argumentation.

Advice for teachers

We researched professional literature for and by teachers. Although the category is small, we found several tips and advice from teachers who write about the salient outcomes of their professional experimentation for teachers who also want to embrace this form of teaching. Category 4 is therefore called "advice for teachers" (**Table 5**). This category helps us to gain more insight into the obstacles that teachers have overcome and the peculiarities they have noticed compared to 'regular' teaching, and what they want to advise their colleagues.

The first category, *(4.1) on implementing the task*, contains hands-on tips on teaching mathematical folding, like: "Once you've obtained a strip of nine triangles, mark the folds on all the inner segments to make handling easier" (Terrier, 2019). In the second category, *on reflecting on the implemented task (4.2)*, teachers mostly specify what they would do differently when they teach this topic next time, or what they have experienced during the folding lesson:

> "In retrospect, I think it is important to offer the folding problem as open-ended as possible. So only those two questions a) and b), to avoid unnecessary calculations as much as possible. It is important that students come to modelling on their own, make a sketch of the pentagon and put letters at the five vertices" (Jansen, 2014, p. 26).

Category *(4.3), preparation of the folding task*, varies from having enough folding paper in stock and how to arrange the tables in de classroom, but also about pre-knowledge, like: "It is assumed that the binary notation has been discussed and we are working towards an extension on measurement using bicimals" (Brissenden, 1972). In *on encouraging and preparing colleagues to use folding in classroom. (4.4)*, tips are provided on how to prepare colleagues for a folding task and also how to get more colleagues enthusiastic about folding, like: "start with a strong, small team of people passionate about

Table 6. Interview results on mathematical topics taught using mathematical folding

Topics (descriptions in Table 2)	n
1.1 Geometry	12
1.2 Relationships and functions	2
1.3 Mathematical topics arising from folding (origami)	2
1.4 Proof	2
1.5 Other mathematical topics	5
Totals overall	23

Note. Extra topics suggested by experts: stochastics; simple linear formulas through a single fold & n: Number of excerpts referring to topic

mathematics, and then other teachers will be drawn in" (Leroux & Santos, 2009, p. 19).

In this section, we have listed the various categories and subcategories that we found via our grounded research approach. We have provided examples of excerpts from all subcategories to clarify our coding process. Although the four categories are very different in terms of content–from mathematical topics to advice for teachers–collectively, they provide a comprehensive picture of teachers' professional experimentation.

Expert Interviews

In the previous section, we presented the results of our grounded research approach on teacher-authored articles on mathematical origami. As a member validation of the results of teachers' professional experimentation in secondary classrooms acquired through analyzing journal articles written by teachers, we interviewed five experts. All experts agreed with the categories of the codebook.

We have summarized the interview coding results in four tables, which follow the structure of the codebook used for the article excerpts in Table 2 to Table 5. The tables also reflect the order of the interview questions. In the "number of excerpts referring to topic" column, we show how often an interview excerpt matched our codebook subcategories, without exposing these subcategories to the interviewees. The final row of each table summarizes suggestions for adding to the (sub)categories.

Table 6 shows that the experts gave many examples of mathematical topics that can be supported with folding and that most examples, like in the articles, were about geometry. A remark from the interviewees indicates how you can interpret *(1.4) proof* as "folding" proof, that it can be a first step towards a more formal proof: "It doesn't have to be an epsilon-delta proof to be okay." The experts also mentioned examples, such as hexaflexagons, that corresponded to topics from the articles but had too low frequency to appear in Table 6, so they are placed at *(1.5) other mathematical topics*. The extra topics that were mentioned in the last row were already included in the subcategories of the codebook (*(1.5) other mathematical topics, (1.2) relationships and functions*).

Table 7. Interview results on teachers' reasons for using folding

Reasons (descriptions in Table 3)	n
2.1 To actively explore mathematics manually	4
2.2 To visualize mathematics	1
2.3 To ground understanding	1
2.4 To approach mathematics in a non-standard way	8
2.5 To create positive affect	7
2.6 To implement low floor/high ceiling tasks	1
Totals overall	22

Note. Experts suggested additional reasons: fun; learning to recognize patterns; showing the beauty of mathematics; and repeating folds being like a function, subroutine, or program & n: Number of excerpts referring to reason

Table 7 shows that the experts, when asked the open question on reasons for using folding, were primarily engaged in approaching mathematics in a non-standard way, to create a positive effect, and to explore mathematics manually. As an example, the excerpt that we coded as *(2.3) to ground understanding* was:

> "But of course, mathematics is very much in your head […], but if you also create it, to make it retain better … and of course, you can do that, for example, by having them fold a box with optimization. So, by basic folding, the mathematical subject is illustrated differently, and it will retain better."

When we asked for more reasons to teach with folding, after exposing experts to this list of categories, one of the interviewees asked, surprised: "Yes, isn't anyone saying to show the beauty of mathematics as well?". Despite being stated in the articles, it was not mentioned often enough to appear as a separate category. Next to these findings, one of the experts was determined about the importance of repeated folding. Though some articles did mention folding as teaching students an *algorithm* (which is placed at *1.5 other mathematical topics* in Table 2), the expert stated, "So it's kind of like a function or subroutine or a program, but you're doing it multiple times. So, the thing is […], function is pretty fundamental. It is pretty important." The additional reason "fun" is included in code *(2.5), to create a positive effect*. Learning to recognize patterns was already placed at *(1.5) other mathematical topics*.

Table 8 shows that some aspects of the learning process were mentioned spontaneously, and the transition from informal to formal mathematics was mentioned most. One of the experts mentioned teaching new mathematical words *(3.1) to transition from informal to formal*: "It is their language; they still have to learn that ..." And at multiple folds for an optimizing task: "But then you also start making a fold. And another fold. [...] And then you start thinking here it's almost zero, then it

Table 8. Interview results on aspects of a teaching process

Aspects (descriptions in **Table 4**)	n
3.1 To transition from informal to formal	3
3.2 To practice skills.	1
3.3 To elicit prior knowledge	0
3.4 To encourage reflection	1
Totals overall	5

Note. Extra aspects suggested by experts: -

Table 9. Interview results on advice for teachers

Advice (descriptions in **Table 5**)	n
4.1 On implementing the task	9
4.2 On reflecting on the executed task	0
4.3 On preparing the folding task	3
4.4 On encouraging and preparing other teachers with respect to using folding in classroom	1
Totals overall	13

Note. Extra advice suggested by experts: work neatly; prepare; let the students work together; it is not bad if someone does not like folding; fold together with your colleagues to teach them folding, it is hard to learn it from a book & n: Number of excerpts referring to advice

gets bigger. Is that close to zero again, then you already think a little bit ...” *(3.4) to encourage reflection*. Experts did not mention extra aspects, only questions about aspects that were already included in the subcategories, like “Is neatly working incorporated with the skills?” Neatly working is incorporated in subcategory *(3.2), practicing mathematical and 21st century skills*.

Table 9 shows that the experts respond in line with the articles to the open question for advice for other teachers to give tips on how to implement and prepare the folding tasks. However, category *4.2 on reflecting on the executed task* is not mentioned unprompted.

The extra pieces of advice that were given were all divided into the categories of the codebook. The advice that “it’s not bad if someone doesn’t like to fold” was phrased as:

> People must do it [folding] to see whether they can do it or like it. And I do not mind if they do not like it, because I do have my students do it [folding] too, just [to do] something different, and some people just think it’s terrible. See, well, then you know that too, but do know [as a teacher] that some students like this. But mostly do it [folding] if you [a teacher] like it yourself.

This advice fits subcategory *(4.4) on encouraging and preparing other teachers with respect to using folding in classroom.*

This section outlines a member validation of the results of teachers’ professional experimentation in secondary classrooms. The experts agreed with the categories in the codebook and when these categories were presented to them, they suggested many of the underlying subcategories.

CONCLUSIONS AND DISCUSSION

We selected and coded excerpts from 45 professional articles from four countries to study teachers’ professional experimentation with mathematical origami in secondary education. From the coding process four categories emerged, on which experts agreed via a member validation procedure. We use these categories to answer the research question: *What do mathematics teachers report about their professional experimentation with mathematical origami in the secondary education classroom?*

From the four emerging categories of our analysis:

(1) mathematical topics taught using mathematical folding,

(2) teachers’ reasons for using folding,

(3) aspects of a teaching process, and

(4) advice for teachers,

we draw four conclusions:

Conclusion category 1. Mathematical origami is applied across a broad range of mathematical topics and domains in secondary education. We found numerous mathematical topics that can be taught using mathematical origami in secondary education. Though most of these topics are in the geometry domain, as we expected, we also found a richness of topics within this domain, ranging from lines to complicated spatial figures. Next, we found other subcategories, such as relationships and functions, mathematical topics that arise from folding and statistics, and the domain of proof, where folding tasks are used to introduce a general principle, eventually requiring proof. We have indicated in Appendix 1 which mathematical topics are covered per article of the 45 articles used. Many articles cover multiple subcategories, including the subcategory proof, like (Cundy, 1985; Pietsch, 2007). So, teachers use mathematical origami in various topics to teach students mathematics that might eventually lead to proof.

Conclusion category 2. Teachers report a variety of reasons for using mathematical origami in the classroom. Teachers give many examples, advice, and justifications for using mathematical origami in class. In doing so, they use words and articulations that we recognize as part of well-known teaching strategies like inquiry-based learning, such as the example of van Oord (2004), where students fold boxes and then discover that there are different contents.

When we make a graphical representation of category 2 (see **Figure 10**), we can rearrange some subcategories according to the match with and *starting at the base*.

What we find noteworthy is that category 2 mentions characteristics of embodied mathematics learning (Gerofsky, 2015), a still actively developing theory. Gerofsky (2015) discusses embodied mathematics learning that challenges mathematics educators to turn

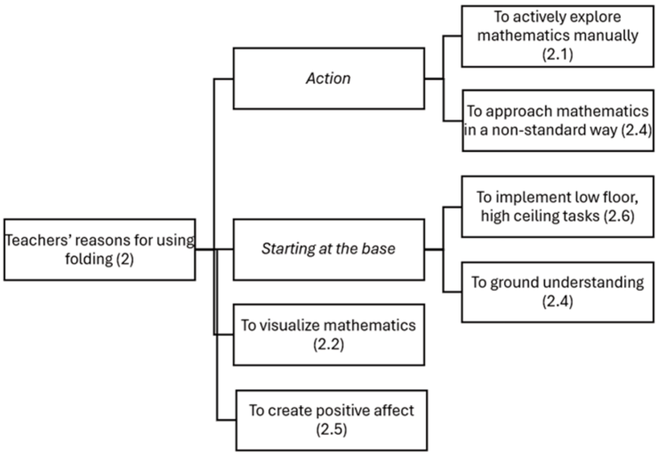

Figure 10. Rearranging the subcategories of category 2: Teachers' reasons for using folding (Source: Authors' own elaboration)

to bodily and sensory experiences as foundational and essential for mathematical sense-making. This can occur with students' engagement with actual physical objects, manipulatives, and other multimodal mathematical work (Gerofsky, 2015). The subcategories (2.1), (2.4), and (2.2) align with arguments for embodied mathematics learning. Research on the effect in mathematics education indicates that learning mathematics is strongly linked to positive emotion, e.g., (Chamberlin, 2010; Hannula, 2014). Subcategory (2.5) aligns with these prompts from research.

Conclusion category 3. Teachers report that mathematical origami can be used for a range of learning objectives in various lesson phases. Mathematical origami is used for distinct phases of the learning process. This is not only at the beginning of the lesson but also throughout the duration of the mathematical task. We noticed that teachers often start a lesson with a mathematical origami folding task, but the process of (re)folding, or the reflection on the folded work itself (manipulative), is then used in various ways, such as learning vectors in 3D (Casteli & Trahe, 2016).

Therefore, it is recommended that future research and teacher training focus on how to teach with mathematical origami, which phases of the lesson it can be integrated into, and the learning objectives that can be set. Further research is also needed to develop guidelines for lesson planning in order to better leverage the potential of mathematical folding activities.

Conclusion category 4. Authors encourage other teachers to pursue classroom folding and have good advice for implementation, but sometimes trivialize how challenging classroom folding can be. How the teacher-authors report on advising other teachers on the use and introduction of mathematical origami sometimes seems to lack some guidance, as it is only a small category in the articles. The total number of excerpts in **Table 5** is 45, compared to 219, 145, and 104 in the other three tables. Between category 3, aspects of a learning process, and category 4, we see overlap in the subtopic *reflection*, so several suggestions were given on how reflection with mathematical origami can be stimulated. The experts interviewed were clearer in explaining what is involved in teaching mathematics using mathematical origami, that introducing it in a classroom can be complicated, and that, as a teacher, you must be prepared. They advised practicing it together with a colleague and actively helping them with it, (**Table 9**), indicating that folding is not something that can easily be learned autonomously. If we make a graphical representation of this category 4 (see **Figure 11**), we note that the recommendations have different timings: before class (4.3), during class (4.1, 4.2) and not time-bound (4.4).

Looking at the four categories, the reasons teachers report for using mathematical origami in the classroom are all related to enhancing students' learning and increasing their enjoyment of lessons. It seems that these teacher-authors are willing to put extra effort into teachers to put extra effort into teaching a mathematical origami lesson and based on the low number of remarks in **Table 5**, they may downplay the challenges involved in teaching with mathematical origami.

Additionally to these conclusions, it's noteworthy that the teachers do not address that mathematical origami would be too childish for secondary education. In contrast, Friedman (2018) points out that mathematical folding is sometimes embraced by mathematicians, and its mathematical potential is praised, but on the other hand, folding is considered too childish, and too volatile, as part of mathematics is in the motion of folding.

This study demonstrates that teachers engage in professional experimentation, are willing to put in extra

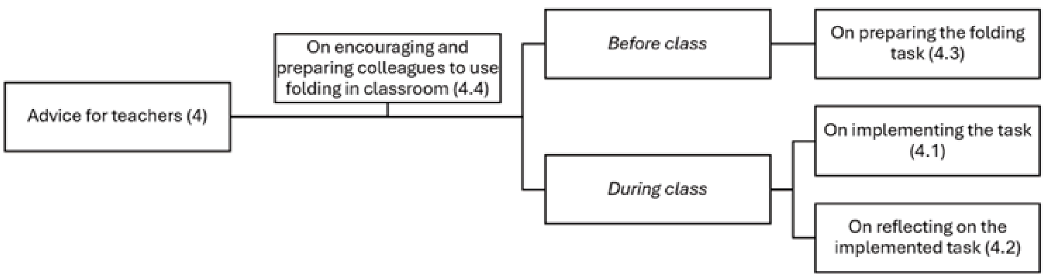

Figure 11. Rearranging category 4: Advice for teachers (Source: Authors' own elaboration)

effort and report on their experiences. Writing about their experiences means inspiration for colleagues and contributes to their own professional growth. Our observations align with Clarke and Hollingsworth's (2002) model of professional experimentation of teachers, as teachers report on what they have learned and experienced and in what phase of the lesson it can be used in class (**Table 4**) and give suggestions on how things go well or can be improved (**Table 5**). They motivate why they are experimenting (**Table 3**) and indicate which topics they are experimenting with (**Table 2**). As teachers report on their experimentation, they enact on the external domain, arrow 5 in this model (see **Figure 1**), which indicates that teachers report to inform other teachers.

Four subdivisions for teaching with mathematical origami emerged from the theoretical analysis. Though it was not our goal to classify articles according to these subdivisions, we noticed that all four subdivisions occurred in **Table 2** on mathematical topics. We can use this classification of teaching mathematics with mathematical origami for our future research to inform teachers about the use of folding tasks that follow from the various subdivisions.

We realize we have chosen a culturally biased sample by choosing articles from GE, FR, the UK, and the NL. We have sketched a first picture of the professional experience of these teachers with mathematical origami, and it appears that they autonomously shape their learning process by experimenting in the domain of practice and reflecting on the salient outcomes. As we have been able to research professional literature from the external domain without questioning the authors themselves, we have observed the teachers' experimentation without interference. The results of this research help us considerably to continue our research with teachers using mathematical origami in secondary education for improving the teaching and learning of mathematics.

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Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

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