MODESTUM EURASIA Journal of Mathematics, Science and Technology Education, 2025, 21(4), em2619
ISSN:1305-8223 (online)
OPEN ACCESS Research Paper https://doi.org/10.29333/ejmste/16187

Mathematical connections promoted in multivariable calculus' classes and in problems-solving about vectors, partial and directional derivatives, and applications

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Received 04 November 2024 ▪ Accepted 09 February 2025

Abstract
In a vector calculus course, the mathematical connections made by an in-service teacher and his engineering students in problems-solving involving vectors and partial and directional derivatives were explored. This study is relevant due to the difficulties in connecting multiple representations and meanings of ordinary and partial derivatives. Networking between the extended theory of connections and the onto-semiotic approach was used. The qualitative methodology included three stages: (1) selection of participants (in-service teacher and students), (2) data collection in four moments: design of the class on partial and directional derivatives and then, the development of this applying participant-observation and recording, design of a questionnaire and its application to the students, and (3) data analysis using theoretical tools. The results showed that the in-service teacher used various connections, starting with the instructional oriented and then others such as meaning, procedural and representations. Students defined and represented vector, partial and directional derivatives concepts, activating meaning connections and different representations. Also, they solved tasks using different connections (different representations, procedural, feature) to find partial and directional derivatives, gradient, curl and divergence. This analysis was carried out in terms of mathematical practices, processes, objects and semiotic functions. 72% of the students gave meaning, represented and appropriately used the concepts of vector calculus, while 28% had difficulties, especially in the procedural connection to find partial derivatives.

Keywords: mathematical connections, vector, partial and directional derivative, vector calculus, teacher, students

INTRODUCTION

Calculus is a relevant area of mathematics with numerous connections within its own concepts and with other subjects, especially, it illustrates the versatility and importance of mathematics in other sciences, engineering, economics, among others (Fuentealba et al., 2018; Park et al., 2015; Rodríguez-Nieto et al., 2024; Yu et al., 2023). Some key connections in calculus focus on the following.

1. The relationship between differentiation and integration as inverse processes, emphasizing the fundamental theorem of calculus (García-García & Dolores-Flores, 2019).
2. Relationships between a powerful set of tools to understand and analyze geometric shapes, symbols and their properties. For example, derivatives can be used to find slopes of curves, tangent lines, and rates of change, while integrals can be used to find areas under curves and volumes of solids of revolution.
3. Vector calculus extends calculus to multivariable functions, connecting vector fields, integrals, curl, divergence, and curvature (Borji et al., 2024). It is

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Contribution to the literature

- This research contributes to the science and teaching and learning processes of vector calculus through the use of mathematical connections when engineering students solve problems related to vector, partial and directional derivatives.
- Also, the special role of mathematical connections used by the in-service teacher when solving problems with partial derivatives is presented, encouraging student participation.
- The onto-semiotic analysis of mathematical connections presented in this article is important because it shows details in mathematical practices, configurations and semiotic functions (SFs) in problems-solving with partial derivatives, curl and divergence.

fundamental in electromagnetism, fluid dynamics, and engineering applications.

It is recognized in research on univariate calculus that students, pre-service mathematics teachers, and in-service mathematics teachers (IMT) face difficulties in connecting the meanings of derivatives and their representations, which impacts concepts such as the instantaneous rate of change (Kertil & Gülbağcı Dede, 2022; Pino-Fan et al., 2018) and graphical transitions between derivatives and antiderivatives (Borji et al., 2018; Fuentealba et al., 2018; Martínez-Panell et al., 2015). Some authors (Font, 2000; Galindo-Illanes & Breda, 2024; García-García & Dolores-Flores, 2021; Rodríguez-Nieto et al., 2023a, 2023c) highlight difficulties in deriving f' from the graphs of f due to reliance on algebraic representations and tabular procedures, which complicates graphical visualization and interpretation.

Studies reveal that mathematics and engineering students struggle with understanding and applying derivatives (Ikram et al., 2020). Bingolbali et al. (2007) noted that engineering students often view derivatives as rates of change, while mathematics students focus on their geometric interpretation as slopes of tangent lines. Zandieh et al. (2000) highlighted students' fragmented knowledge of derivatives, lacking awareness of their conceptual connections. Recent research emphasizes persistent difficulties, suggesting the need to address these challenges through enhanced meanings, representations, arguments, and application problem-solving strategies (Font & Rodríguez-Nieto, 2024; Galindo-Illanes et al., 2024).

Research on multivariable calculus highlights the complexity of partial derivatives, emphasizing the need to teach connections between concepts. Martínez and Vinuesa (2022) found these links challenging for first-year students, particularly when exploring real functions of two variables graphically, underscoring the importance of fostering understanding in calculus. Furthermore, "it is common, for instance, for them to think that the existence of partial derivatives $f_x(a,b)$ and $f_y(a,b)$ implies that the plane $z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$ is a tangent plane to the surface $z = f(x,y)$ at $(a,b,f(a,b))$ and so f is differentiable at (a,b)" (Martínez & Vinuesa, 2022, p. 733). Also, emphasis should be placed on the geometric representations of partial derivatives where students must identify and imagine the planes to $x = c$ and $y = c$, as well as the surface of the curve (Wangberg & Johnson, 2013; Wangberg et al., 2002).

Thompson et al. (2006, 2012) highlighted students' difficulties in connecting physical context with mathematics, particularly in interpreting meanings of partial derivatives in physics. To address this, activities with a didactic-mathematical approach were designed to enhance understanding of partial derivatives in thermodynamics and graphically interpret mixed partial derivatives (Bajracharya et al., 2019; Thompson et al., 2012). Calculus should go beyond operations and graphs, adopting a STEM approach that integrates symbolic, graphical, verbal, and numerical representations in physics. Poor understanding of derivatives and their applications underpins these efforts. In thermodynamics, partial derivatives are often confusing, as students struggle to relate small numerical differences to solve problems effectively (Dray et al., 2019; Roundy et al., 2015).

Martínez-Planell et al. (2015) interconnected various concepts of multivariable calculus (slope, tangent plane, partial and ordinary derivatives, directional derivative, etc.) through local linearity, using genetic decomposition from the APOS theory. Moreno-Arotzena et al. (2021) emphasized how different representations and their interconnections enhance the learning of gradients in functions of two variables. Additionally, understanding covariational reasoning in rate of change is crucial, as it extends beyond two-variable functions to directional derivatives (Weber, 2015; Weber et al., 2012). McGee and Moore-Russo (2015) showed that students grasp partial derivatives and vector calculus concepts better when they consider the slope of a line in space as fundamental. Martínez-Planell et al. (2017) confirmed that using tangent planes helps students in learning partial and directional derivatives, improving graphical interpretation in multivariable calculus.

The difficulties of students in problems-solving with partial and directional derivatives, gradient, curl, among others, are also due to the fact that students still continue to present difficulties in understanding the concept of vector, operating with vectors, finding the norm, the vector and scalar product, angles, etc. (Barniol & Zavala, 2016; Flores-García et al., 2007; Possani et al., 2010; Rakkapao et al., 2016; Rodríguez-Nieto et al., 2024; Salgado & Trigueros, 2014; Susac et al., 2018). This problem is not new. Research shows that the vector concept is fundamental yet challenging. University students struggle to define and apply it, as their prior training focused on solving mechanized, non-contextualized problems in calculus, lacking connections with other fields like physics (Flores et al., 2017; Gutierrez & Martín, 2015).

Tairab et al. (2020) identified difficulties in learning the concept of vectors, such as a lack of conceptual understanding and issues with procedures for obtaining vector and scalar products. Students often perform calculations without understanding the connections between magnitude, direction, and quantities, limiting their ability to connect conceptual and procedural aspects. This impacts problem-solving skills, such as finding directed segments or norms. While some excel in vector operations, others fail to integrate conceptual, representational, and procedural aspects, complicating their overall understanding (Cárcamo et al., 2023; Rodríguez-Nieto et al., 2024). It has been recognized that Rodríguez-Vásquez et al. (2024) addressed the concept of vector in the new Mexican school because teachers and students experience significant challenges for their teaching and learning, that, in fact, the difficulties remain when students must perform operations and representations.

After reviewing the literature on the importance of calculus in one variable and several variables, it was found that it is an important branch of mathematics for students, mathematics teachers and is used in other careers such as engineering, economics, physics, chemistry, etc. However, due to the complexity of the mathematical objects involved in vector calculus or in several variables, students have difficulty graphing vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, perform operations with vectors, perform ordinary and partial derivatives, apply the chain rule for composite functions, and solve application problems. These difficulties occur due to the various errors that students make and that can be caused by not making necessary mathematical connections where representations, procedures, and definitions are involved in the resolution of extra-mathematical problems (Cárcamo et al., 2023; Rodríguez-Nieto et al., 2024). Therefore, the research aim is to explore the mathematical connections that a mathematics teacher and his engineering students activate when problems-solving about vectors, partial derivatives and applications.

THEORETICAL FRAMEWORK

Extended Theory of Connections

In this theory, a mathematical connection is understood from view of the extended theory of connections (ETC) and the onto-semiotic approach (OSA) as the tip of an iceberg made up of a conglomerate of practices, processes/objects (problem situations, languages, procedures, propositions, definitions, and arguments), and SFs that relate them (Rodríguez-Nieto et al., 2022a). The connections are important for mathematical understanding. Likewise, mathematical connections can be intra-mathematical "are established between concepts, procedures, theorems, arguments and mathematical representations of each other" (Dolores-Flores & García-García, 2017, p. 160), and extra-mathematical connections, which "establishes a relationship of a mathematical concept or model with a problem in context (not mathematical) or vice versa" (Dolores-Flores & García-García, 2017, p. 161). Extra-mathematical connections are based on intra-mathematical connections and are important for students and in-service teachers in problems-solving in the classroom (De Gamboa et al., 2020). Mathematical connections are one of the mathematical processes that foster mathematical creativity (Sánchez et al., 2022). Each of the mathematical connections of the ETC is described below (**Figure 1**).

1. **Modelling:** It refers to the relationship that a person establishes between the world of mathematics and the real world (or the daily life of students) and between mathematics and other sciences. It can be understood as the connection formed between a mathematical concept and a real-world task (either occurring or potentially occurring in everyday life) or a practical application in a field outside of mathematics. In this process, the subject constructs a mathematical model based on the task of finding a solution. When the subject builds the mathematical model, he uses various knowledge (mathematical or not) by executing multiple actions (algebraic, symbolic, graphic, etc.) to reach an answer consistent with the requirement posed (Campo-Meneses & García-García, 2023; Dolores-Flores & García-García, 2017; Evitts, 2004).
2. **Instruction-oriented:** It refers to the understanding and application of mathematical concept D derived from two or more related concepts, B and C. These connection types can be recognized in two forms:

   (1) the relationship of a new topic with previous knowledge, and

   (2) the mathematical concepts, representations, and procedures connected are considered

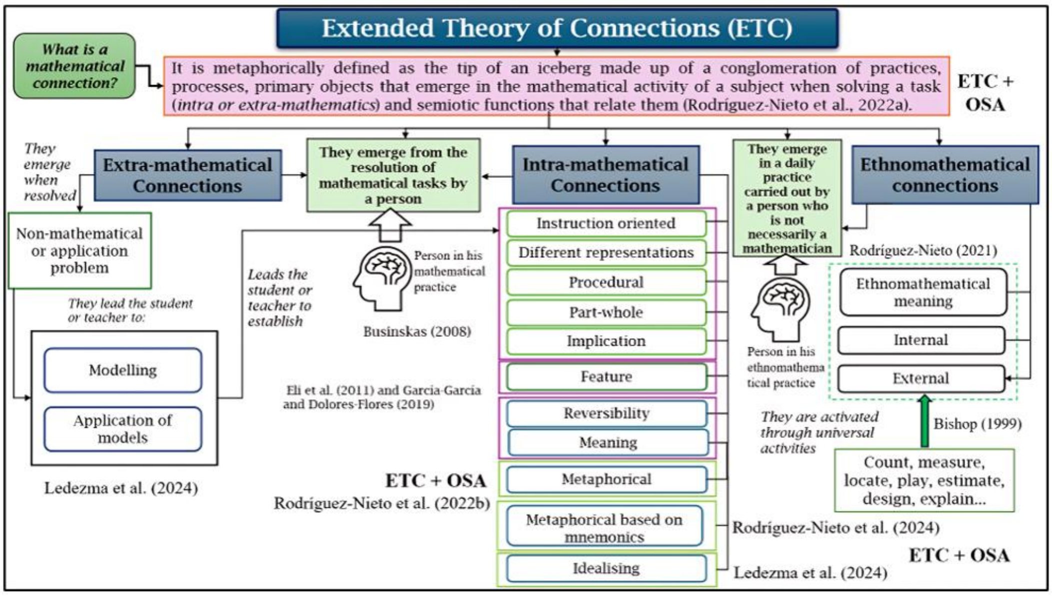

Figure 1. Synthesis of ETC (Rodríguez-Nieto et al., 2024)

fundamental prerequisites that people must have to develop new content (Businskas, 2008).

For example, when the teacher explains to the students that, in order to work on the partial derivative of a function, they must first recall the concepts of functions, limits, the global derivative, and the slope of a line.

3. **Procedural:** This connection is one of the most used by people and is evident when rules, algorithms, or formulas are used to arrive at a result (García-García, 2019). For example, to find the solutions of a quadratic equation $\boldsymbol{ax^2 + bx + c = 0}$ you can use the formula $\boldsymbol{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$.
4. **Part-whole:** This connection is identified when it is institutionally assumed that A is a generalization of B, where B is a particular case of A. For example, the function $\boldsymbol{P(x) = 3x^3 - 2x^2 - 7x - 5}$ is a particular case of the general expression $\boldsymbol{f(x) = ax^3 + bx^2 + cx + d}$. These relationships can be of inclusion when a mathematical concept is contained in another (Businskas, 2008).
5. **Implication:** This connection is based on a logical relationship if-then $(P \rightarrow Q)$ (Businskas, 2008; Mhlolo, 2012). If a function $\boldsymbol{f}$ is increasing on an open interval (a, b), then $\boldsymbol{f'}$ is positive on that same interval.
6. **Different representations:** These mathematical connections can be alternate or equivalent. It is alternate if a person represents a mathematical concept in two or more different ways in different registers of representation: graph-algebraic, verbal-graph, etc. (Businskas, 2008). For example, an alternate representation is shown in Figure 2, where the vector $\boldsymbol{\vec{R} = 11i + 8j + 10k}$ graphed.

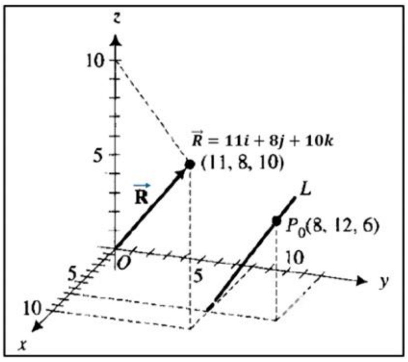

Figure 2. Connection between different representations (alternate) (Leithold, 1998, p. 828)

While an equivalent representation involves a transformation within the same register, such as algebraic to algebraic, graph to graph, or symbolic to symbolic. For example, $\boldsymbol{\vec{R} = 11i + 8j + 10k}$ is equivalent to $\boldsymbol{\vec{R} = \langle 11, 8, 10 \rangle}$ in the algebraic or symbolic semiotic register.

7. **Feature:** It is identified when the person manifests some characteristics of the concepts or describes its properties in terms of other concepts that make them different or similar to others (Eli et al., 2011; García-García & Dolores-Flores, 2019). For example, when the person mentions some elements of a polynomial function $\boldsymbol{f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0}$ (derivative function or antiderivative function) are coefficients (all, $\boldsymbol{a_i}$, with $\boldsymbol{i = 0, 1, 2, 3, \ldots, n}$), literal or variables (in this case, the "$\boldsymbol{x}$") and exponents of the variables $(\boldsymbol{n, n-1, n-2, \ldots, 1})$.
8. **Meaning:** This connection is activated "when students attribute a meaning to a mathematical concept as long as what it is for them (which makes it different from another) and what it represents; it can include the definition that they

have built for these concepts" (García-García, 2019, p. 131). Likewise, students or teachers express what the mathematical concept means to them, including their context of use or their definitions (García-García, 2019, p. 131). We recognize the existence of a mathematical connection between meanings, which is also activated when a person applies these meanings to solve a problem. For example, "the derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$" (Stewart, 199, p. 153).

9. **Reversibility:** It is present when a student or teacher starts from a concept P to get to a concept Q and invert the process starting from Q to return to P (García-García & Dolores-Flores, 2021). This connection is activated when the fundamental theorem of calculus is used to link both concepts, and when the person establishes a bidirectional relationship between the derivative and the integral as operators.
10. **Metaphorical (MT):** It is understood as the projection of the properties, characteristics, and other aspects of a known domain to structure a less familiar one. For instance, when a teacher or student uses verbal expressions like "travel through the graph without lifting the pencil from the paper", they implicitly evoke the conceptual metaphor "the graph is a path" (Rodríguez-Nieto et al., 2022b).
11. **MT connections based on mnemonics:** This connection is "understood as the relationship established by the subject between a mnemonic rule (often a familiar resource) and a mathematical object, rule, or mathematical procedure to memorize and use strategically more easily" (Rodríguez-Nieto et al., 2024, p. 18). These types of connections are both inclusive and recursive, with three key elements to consider:
    (1) keywords that are similar to the word (or term) being referenced,
    (2) acronyms, which are formed when the first letter of each word in a list is used to construct a new word, and
    (3) acrostics which consist of constructing a sentence, where the first letter of each constitutes the term studied (Mastropieri & Scruggs, 1989; Rodríguez-Nieto et al., 2024).
12. **Idealizing:** This connection relates an ostensive to a non-ostensive. Its function is to dematerialize the ostensive and turn it into an ideal mathematical object (for example, the bottom of a rounded tank is considered circle/circumference) (Ledezma et al., 2024).

This theory of mathematical connections can continue to be extended and the work of Cantillo-Rudas et al. (2024) stands out, where they report new developments on neuro-mathematical connections associated with the cognitive part of this theory and the assessment of the brain areas activated in a person's mathematical activity.

Onto-Semiotic Approach

The OSA is a theoretical approach that has impacted research in mathematics didactics due to its special tools to improve the teaching and learning processes of mathematics that involve geometry, statistics, calculus, textbook analysis, didactic suitability, etc. In addition, it assesses the person's mathematical knowledge considering epistemology, anthropology and the ontology of mathematical objects. This approach arose from the need to clarify, connect and improve theoretical and methodological notions of various theories and it is essential to describe mathematical activity from an institutional or personal perspective, modeled in terms of practices and configuration of objects and primary processes activated in said practices (Drijvers et al., 2013).

Mathematical practice is understood as "any situation or expression (…) carried out by someone to solve mathematical problems, communicate the solution obtained to others, validate it or generalize it to other contexts and problems" (Godino & Batanero, 1994, p. 334). This practice includes objects used in a broad sense to refer to any entity that is involved in mathematical practice and is identified as a unit (Font et al., 2013). Furthermore, six main objects are considered:

(1) problem situations,
(2) languages,
(3) definitions,
(4) propositions,
(5) procedures and
(6) arguments.

These interconnected objects form the configuration of primary objects (Godino et al., 2019).

In mathematical practice, primary objects emerge in various ways, shaped by different modes of seeing, speaking, operating, and more. This diversity enables us to categorize them as personal or institutional, ostensive or non-ostensive, unitary or systemic, intensive or extensive, and content or expression. Now, a configuration is a heterogeneous set or system of interrelated objects, which can be institutional (epistemic) or personal (cognitive) (Godino et al., 2019).

The set of primary objects arises in mathematical activity through the activation of fundamental mathematical processes (communication, problem setting, definition, enunciation, procedures (algorithms) and argumentation to justify the procedures) triggered by the application of the process-product vision to said primary objects, which occur together with those

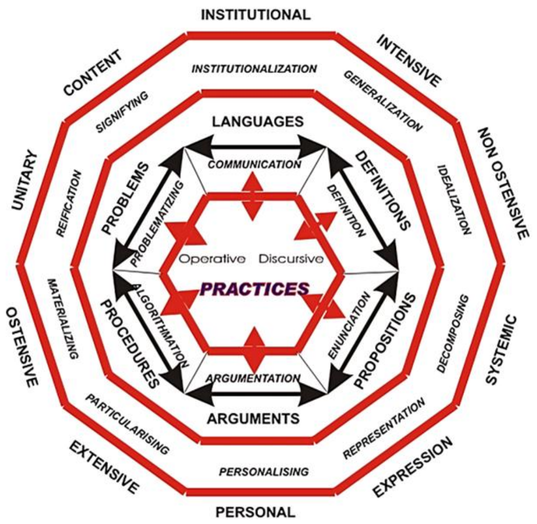

Figure 3. The schematization of mathematical knowledge from an onto-semiotic view (Font & Contreras, 2008).

derived from applying the process-product duality to the five dualities (institutional/personal, ostensive/non-ostensive, expression/content, extensive/intensive and unitary/systemic): personalization-institutionalization, materialization-idealization, representation-meaning, synthesis analysis, and generalization-particularization (Font et al., 2013; Godino et al., 2007), see **Figure 3**.

In the OSA, SF is also considered fundamental, which gives the opportunity to associate practices with activated processes and objects and the construction of an operational notion of knowledge, mathematical understanding, meanings and competencies and other primary objects (Godino et al., 2007). An SF is a triadic relationship between an antecedent (expression/object) and a consequent (content/object) carried out by a person (person or institution) according to a specific criterion or correspondence code (Font, 2007), establishing not only unidirectional but bidirectional relationships between two objects. SFs are inferred when mathematical activity is seen from the expression/content duality, among others. The notion of SF (OSA) is more general than the notion of mathematical connection (ETC) since connections are considered particular cases of SF of a personal or institutional nature. In the ETC, the mathematical connection can be true or not, revealing from the perspective of OSA that when a subject makes a correct connection, it coincides with the institutional one, while when the connection is incorrect, it is personal (Rodríguez-Nieto et al., 2022a).

Godino emphasized the importance of addressing the meanings of mathematical objects by individuals in specific contexts, highlighting the semiotic-cognitive issue, which is connected to knowledge defined in the OSA as the "set of" relationships that the subject (person or institution) establishes between objects and practices, relationships that are modeled through the notion of SF (Godino, 2022, p. 8).

In this research, the ETC-OSA networking is used because the OSA allows for detailing mathematical connections in terms of practices, processes, objects, and SFs that relate them, which facilitates a deeper understanding of a topic related to a specific mathematical concept. Additionally, the ETC enables the use of typologies of mathematical connections. For this reason, this networking serves as a special framework for analyzing mathematical connections, which evolves and refines its tools as the research progresses (Rodríguez-Nieto et al., 2024).

Synthesis of the Networking ETC-OSA

Establishing a networking between two or more theories allows us to investigate and understand how their contributions can be successfully linked (or not), while respecting the conceptual and methodological principles that support them. Likewise, it helps to understand and detail the complexity of the phenomena involved in the teaching and learning processes of mathematics (Kidron & Bikner-Ahsbahs, 2015; Prediger et al., 2008). This research draws on the work of Rodríguez-Nieto et al. (2022a), who integrated the ETC and OSA frameworks. Their study focused on three aspects: the nature of mathematical connections in each theory, the exploration of connections in subjects written and verbal productions (operational and discursive), and a content analysis of key publications from both theories to identify principles, methods, and research questions. They also examined whether there are congruencies and complementarities between the ETC and OSA to broaden the understanding of connections. A detailed analysis of mathematical connections followed the integration method proposed by Drijvers et al. (2013), Kidron and Bikner-Ahsbahs (2015), and Radford (2008), which involved selecting and describing episodes to enhance the identification of mathematical connections using the ETC and OSA.

In Rodríguez-Nieto et al. (2023b) data were analyzed in terms of practices, configurations of primary objects and SF that relate them as proposed by the OSA. Subsequently, parts of mathematical activity (e.g., practices, primary objects, and science fiction) were encapsulated as a type of connection proposed in the ETC (meaning, feature, procedural, part-whole, ...). It should be noted that, although the analysis methods are different (thematic analysis for the ETC and analysis of mathematical activity for the OSA), the main conclusion is that both theories work together to offer a deeper understanding of mathematical connections. The analysis conducted with OSA tools reveals a mathematical connection as the tip of an iceberg, metaphorically speaking. This "tip" consists of various practices, processes, primary objects activated in these practices, and related SFs. Through this approach, the structure and function of the connection are thoroughly examined.

Figure 4. Some study participants (Source: Field study)

In essence, OSA tools help uncover the complexities behind the seemingly simple mathematical connections, providing a detailed view of the interrelated elements that contribute to their formation (Rodríguez-Nieto et al., 2022a).

This theoretical framework that emerges from the articulation between ETC and OSA is not only applied to concepts of differential calculus, but also to concepts from other subjects such as vector calculus (Rodríguez-Nieto et al., 2024), ethnomathematical connections and geometry (Rodríguez-Nieto et al., 2023c), differential equations (Dans-Moreno et al., 2022), ethnomathematics and STEAM (Rodríguez-Nieto & Alsina, 2022). In this study, we will work with vectors, partial derivatives and applications that involve the chain rule, curl and divergence.

METHODOLOGY

This research is qualitative (Cohen et al., 2018) and developed in three stages: the first refers to the selection of study participants (a teacher and his students). The second refers to the data collection where four moments (m) were considered.

1. **m1:** Design of a class on partial derivatives by the in-service teacher.
2. **m2:** The teacher develops the class with his students, encouraging participation and use of the blackboard (applying participant-observation and recording).
3. **m3:** Design of a questionnaire that involves vectors, partial and directional derivatives, curl and divergence.
4. **m4:** Application of the questionnaire to students.

The third stage refers to data analysis using the theoretical and methodological tools that emerged from the ETC-OSA networking (Rodríguez-Nieto et al., 2022a).

Participants and Context

The participants in this research were: an IMT of vector or multivariate calculus who was 29 years old and had 10 years of work experience (with 5 years teaching calculus in higher education). This IMT has a degree in mathematics, a master's degree and a PhD in mathematics education and focuses on calculus. He currently works at a private university on the Colombian Caribbean Coast. The other participants were 202 engineering students (P1, P2, P3, …, P202) in their third semester (of ten required) enrolled in the subject of vector calculus (developed by the participating professor) from the same university (**Figure 4**).

It should be noted that the students had already taken and passed the differential and integral calculus subjects where they developed topics such as ordinary derivatives, limits, functions, integrals, etc., which are important to successfully complete calculus in several variables. These students come from the city of Barranquilla and nearby municipalities or towns and

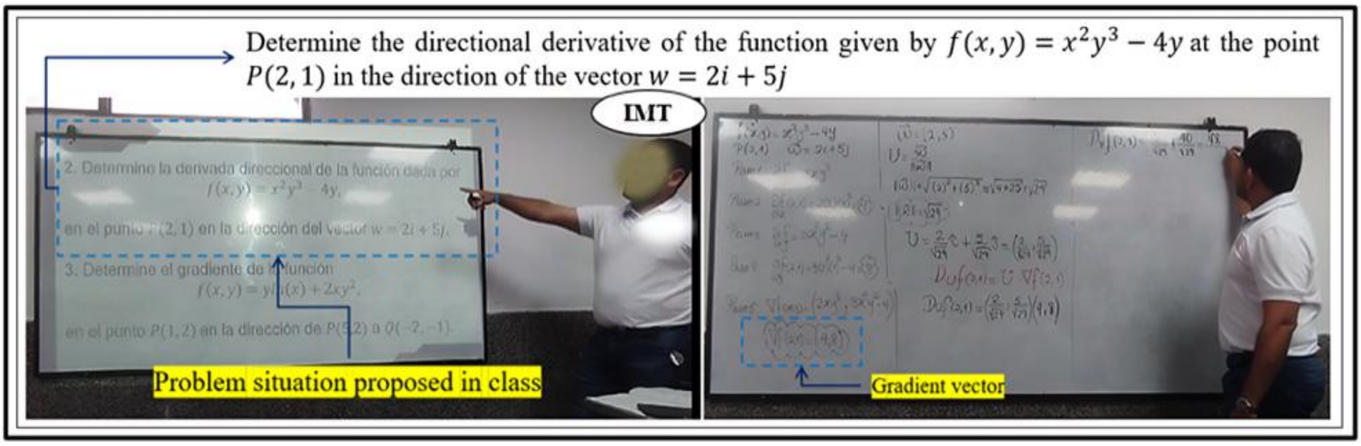

Figure 5. Explanation from the teacher in the classroom (Source: Field study)

Table 1. Questionnaire with tasks

No	Task
Preliminary	*For you, what is a vector and how is it represented? What does the partial derivative mean and how is it represented?* The answers to these questions are shown in the first part of the results section, that is, before the temporal narrative.
1	Calculate the directional derivative for the function $f(x, y) = 2x^2 + 5y^2 + 5x^3 - 8y$ in the direction $U = cos\left(\frac{\pi}{4}\right) i + sin\left(\frac{\pi}{4}\right) j$ and evaluate said derivative at the indicated point $D_u f(0,3)$.
2	Calculate the gradient of w evaluated at the indicated point: $w(x, y) = cos(xy) + sin(x^2y)$ at the point $P = \left(1, \frac{\pi}{4}\right)$.
3	Find an equation of the plane tangent to the surface at the indicated point: $z = e^{3x} sin(3y)$ at the point $P = \left(0, \frac{\pi}{4}, 1\right)$.
4	Calculate the $curl(\vec{F}(x, y, z))$ for the following vector field and evaluate at the indicated point: $\vec{F}(x, y, z) = (x^2 + z^2)i + xe^y \cos(z)\, j - xe^y \cos(z)\, k;\ P = \left(-2, 0, \frac{\pi}{4}\right)$.
5	Calculate the divergence $div(\vec{F}(x, y, z))$ for the following vector field and evaluate at the indicated point: $\vec{F}(x,\ y,\ z) = 2x\, i + 3y^2\, j\ - 5z^3 k;\ P(1,2,-1)$.

their participation was voluntary, who agreed to participate in the project that does not have economic purposes but rather educational ones.

Data Collection

In the second stage referring to the collection of information, at m1 it is evident that the IMT prepares his class considering the topics suggested in the syllabus of the subject of vector calculus and textbooks such as Leithold (1998). Also, the IMT in his class uses PowerPoint slides to address definitions, theorems and propose problems and considers it important that, in solving the problems, different representations, propositions, definitions, arguments, etc. are activated. Furthermore, at m2 the researchers video-recorded (applying participant-observation) some episodes of the classes where the IMT explains topics about vectors, partial derivatives and directional derivative and gradient vector (**Figure 5**).

It should be noted that the IMT encourages participation in the classroom so that students solve problems on the blackboard, interact with each other and with the teacher, considering it important for the participant to carry out step-by-step procedures. Now, if at any m a student proceeds incorrectly, the other students intervene to help him or the teacher states which connection has not been activated and based on the error, establishes the mathematical connection that leads to an appropriate procedure.

Subsequently, at m3, a questionnaire was designed based on the topics addressed by the IMT, starting with a task about knowledge about vectors, partial derivatives (they were not considered for qualification the students' questionnaire) and the problem-solving of directional derivative, curl and divergence (**Table 1**), considered for qualification.

The criteria for creating the tasks proposed in **Table 1** are directly influenced by the problems identified in the literature review in the Introduction section and the difficulties that students present during classes. For example, in the preliminary task, the meaning of the partial derivative and its representations are promoted. In task 1, the directional derivative is determined by connecting concepts such as function in several variables, partial derivatives, unit vector, among others. In task 2, the importance of the gradient vector connected to the nabla vector, whose components are partial derivatives, is observed. Once again, in task 3, the emphasis is on students solving a problem on tangent planes that involves functions, partial derivatives, planes, tangents, slopes, among other concepts related to

Table 2. Phases for analyzing data based on ETC integrated with OSA

No	Phases	Description
1	Transcription of class observations, interviews, and data organization	The teacher and students' written work was arranged in a way that allowed the researchers to become acquainted with their responses. Furthermore, this step is essential to guarantee that the researcher thoroughly examines, reads, and interprets the gathered information.
2	Temporal narrative	The student's solution to the task is explained mathematically. It includes the actions performed by the student and highlights key primary objects identified in the narrative. The same is done with the in-service mathematics teacher.
3	Mathematical practice	The mathematical practices, or series of actions, are outlined and governed by rules established by institutions that are helpful for problem-solving. Within these practices, the underlying foundation of each mathematical connection in the ETC is clear.
4	Cognitive configuration	It is the system of primary mathematical objects that a subject mobilizes as part of the mathematical practices developed to solve a specific problem. In fact, these primary objects are a fundamental part of the connection because they are generally the beginning (antecedent) and end (consequent) of its structure.
5	Semiotic functions	SFs are established between the primary objects of the cognitive configuration. The mathematical connections suggested by the ETC are established and represented.
6	Mathematical connections	Report of mathematical connections from an articulated ETC-OSA view.

calculus. Task 4 and task 5 were proposed to solve application problems by finding the curl and divergence of vector functions.

These tasks take place in the subject of vector calculus, especially to develop unit 2 referred to the differential calculus of functions of more than one variable, whose learning result is: to use differential calculus in several variables to solve problems that strengthen the professional profile of the engineer. To achieve this result, the following performance indicators are proposed:

(1) identifies the concepts and rules of the derivation of functions in several variables,

(2) correctly explains operations with derivatives in space, and

(3) solves problems of curl and divergence of vector fields.

Finally, these tasks are aligned to achieve the objective of this research.

Data Analysis

The data analysis will be carried out based on the method of analysis of mathematical connections based on the networking between the ETC and the OSA, considering the narrative, mathematical practices, cognitive configuration, SFs and mathematical connections (Rodríguez-Nieto et al., 2024), see **Table 2**.

Below, the results of mathematical connections established by the IMT and the engineering students are reported, following the analysis method presented in **Table 2**.

RESULTS

The results of this research are organized based on the participants' mathematical activity, following the analysis method outlined in **Table 2**. To begin with, an example response for the preliminary task described in **Table 1** is provided. Due to space limitations, a representative example from student P59 (in response to task 1) is highlighted, along with cases and written work from other students who approached the task similarly. Subsequently, examples from students who made errors stemming from disconnections or personal connections are presented

Findings Obtained in the Application of the Preliminary Task

Initially, all students were asked two questions: for you, what is a vector and how is it represented? What does the partial derivative mean and how is it represented? With the objective of knowing what the students were understanding about the fundamental concepts involved in the questionnaire. For example, P11, P24, and P84 recognized that a vector is a quantity that has magnitude, direction and direction and they represented it symbolically in a general way in $\mathbb{R}^n$ and, particularly in $\mathbb{R}^2$ and $\mathbb{R}^3$ (**Figure 6**).

It is worth noting that students P81 and P84 illustrated the vector $\vec{v}$. Furthermore, the students demonstrated an understanding of the partial derivative as the derivative of a multivariable function with respect to one of its variables (**Figure 7**). Notably, students P11, P35, and P59, among others, connected the concept of partial derivatives to their graphical representation. This indicates that the students have at least a foundational understanding of the geometric interpretation of this concept.

In fact, student P90 explains the partial derivative in terms of other concepts such as the rate of change, which shows that he understands the slope of the line contained in the plane tangent to the surface (**Figure 8**).

In the case of P84, other meanings of the partial derivative were evident considering the notion of limit.

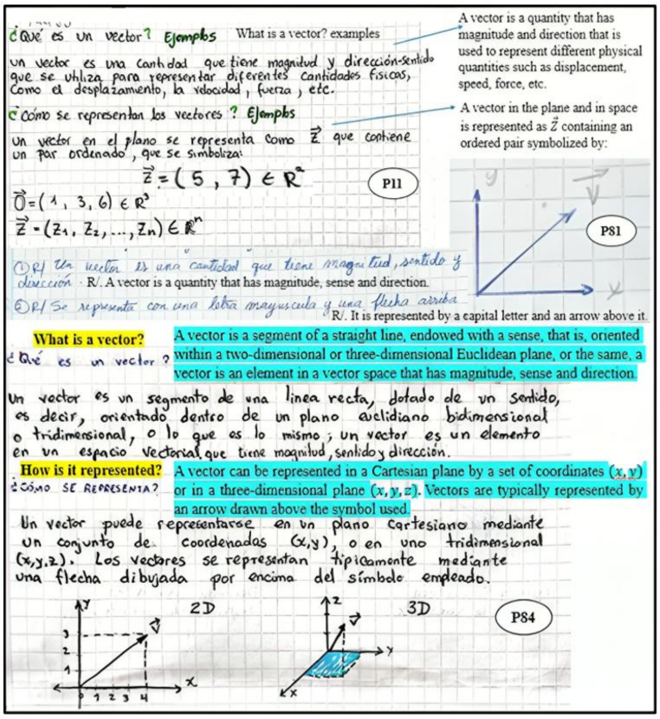

Figure 6. Vector meaning and different representations (Source: Authors' own elaboration)

Temporal Narrative

It will be exemplified with the narrative of P59 when he carried out task 1 of the questionnaire. In this context, P59 read and understood task 1 by identifying the function, the direction of vector U and the point (0, 3).

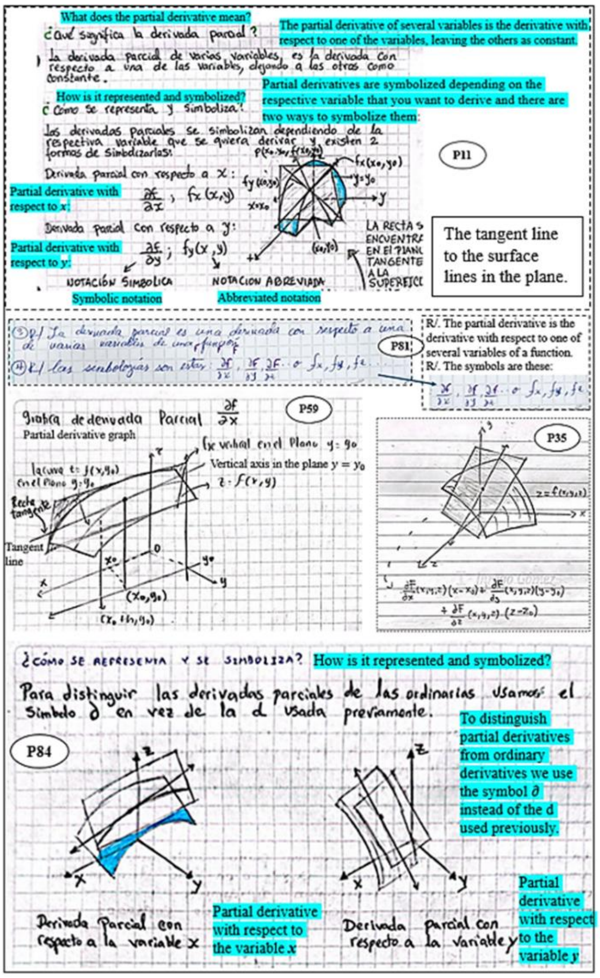

Figure 7. Meaning of the partial derivative and representations (Source: Authors' own elaboration)

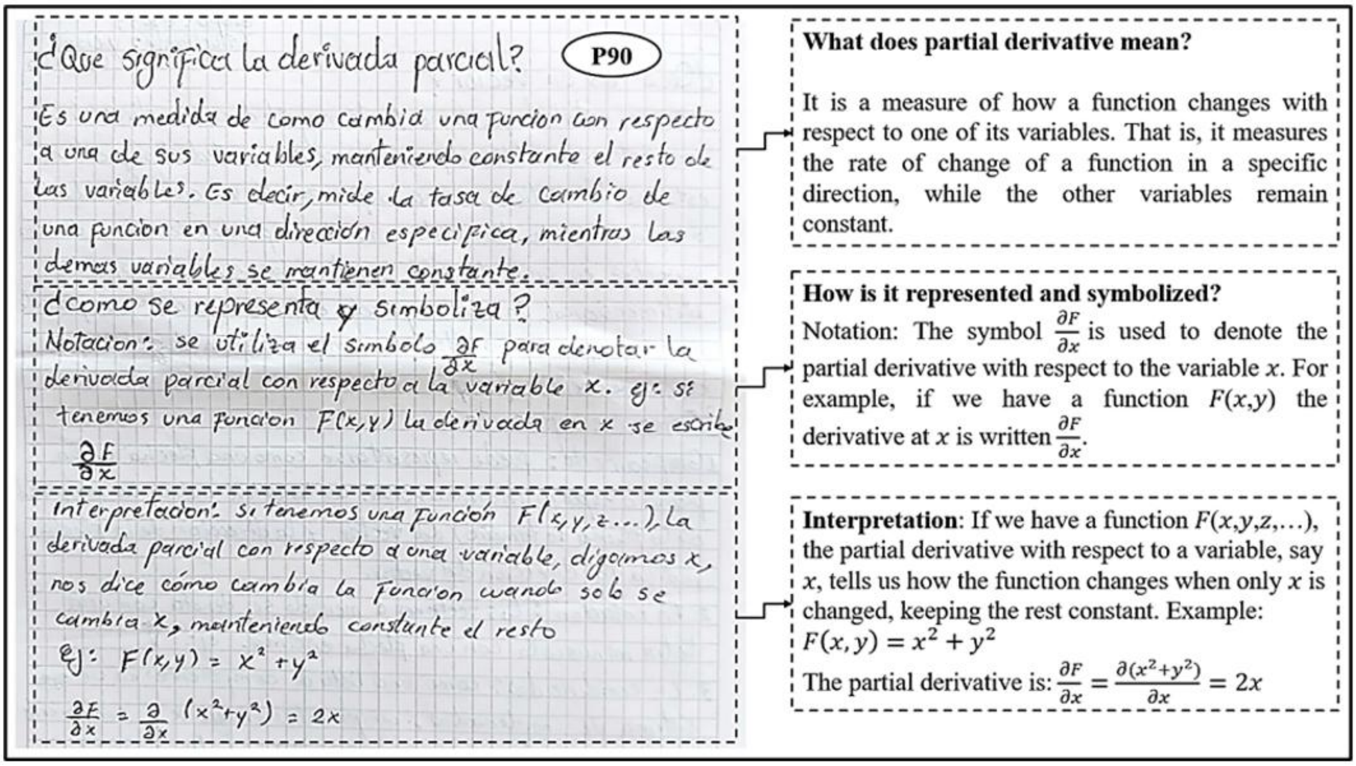

Figure 8. Evidence of meaning, representation and exemplification on the partial derivative (Source: Authors' own elaboration)

He considered the formula $D_U f(x,y) = U \cdot \nabla f(x,y)$ meaning that the directional derivative is equal to the dot product between the unit vector and the gradient of the function. P59 states that the previous formula is equivalent to the following formula: $\boldsymbol{D_U f(x,y)} = (\boldsymbol{cos(\theta)}, \boldsymbol{sin(\theta)}) \cdot (\boldsymbol{f_x(x,y)}, \boldsymbol{f_y(x,y)})$, stating that the gradient is a vector whose components are partial derivatives. Then, perform the dot product $\boldsymbol{D_U f(x,y)} = \frac{\partial f(x,y)}{\partial x}\boldsymbol{cos}\,(\boldsymbol{\theta}) + \frac{\partial f(x,y)}{\partial y}\boldsymbol{sin}\,(\boldsymbol{\theta})$ to find the directional derivative**.** Subsequently, take the function and the angle and substitute it into the formula*:* $\boldsymbol{D_U f(x,y)} = \frac{\partial}{\partial x}(\boldsymbol{2x^2 + 5y^2 + 5x^3y - 8y})\boldsymbol{cos}\left(\frac{\pi}{4}\right) + \frac{\partial}{\partial y}(\boldsymbol{2x^2 + 5y^2 + 5x^3y - 8y})\boldsymbol{sin}\left(\frac{\pi}{4}\right)$. P59 found the partial derivative of the function f with respect to x, obtaining: $4x + 15x^2y$ and found the partial derivative of the function f with respect to y, obtaining: $10y + 5x^3 - 8$, and these derivatives were replaced in the formula. P59 determined the values of $sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and replaces them in the formula: $\boldsymbol{D_U f(x,y)} = (4x + 15x^2y)\left(\frac{\sqrt{2}}{2}\right) + (10y + 5x^3 - 8)\left(\frac{\sqrt{2}}{2}\right)$. Then he applied the distributive property $\boldsymbol{D_U f(x,y)} = \left[(\boldsymbol{4x})\left(\frac{\sqrt{2}}{2}\right) + (\boldsymbol{15x^2y})\left(\frac{\sqrt{2}}{2}\right)\right] + \left[(\boldsymbol{10y})\left(\frac{\sqrt{2}}{2}\right) + (\boldsymbol{5x^3})\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{8}\left(\frac{\sqrt{2}}{2}\right)\right]$ and simplifies it $\boldsymbol{D_U f(x,y)} = \left[(\boldsymbol{2x})(\sqrt{\boldsymbol{2}}) + \frac{(\boldsymbol{15x^2y\sqrt{2}})}{\boldsymbol{2}}\right] + \left[(\boldsymbol{5y})(\sqrt{\boldsymbol{2}}) + \frac{(\boldsymbol{5x^3})(\sqrt{\boldsymbol{2}})}{\boldsymbol{2}} - (\boldsymbol{4\sqrt{2}})\right]$. P59 replaces the point $P = (0,3)$ in the directional derivative $D_U f(0,3) = \left[(2(0))(\sqrt{2}) + \frac{(15(0)^2(3)\sqrt{2})}{2}\right] + \left[(5(3))(\sqrt{2}) + \frac{(5(0)^3)(\sqrt{2})}{2} - (4\sqrt{2})\right]$ and then, applying arithmetic operations, they obtained the following*:* $D_U f(0,3) = 15\sqrt{2} - 4\sqrt{2} = 11\sqrt{2}$.

Using the example of narrative, P59's mathematical practices are developed.

Mathematical Practices

IMT mathematical practices (MpIMT)

Mp1IMT: IMT read and understood the problem proposed with his students, highlighting that he must first find the gradient of the function at $P = (2, 1)$ and said that it is important to know how to derive, recognize the slope, the rate of change as requirements to address the partial derivation.

Mp2IMT: IMT stated that the directional derivative is the rate of change of the function in the direction of a vector.

Mp3IMT: IMT stated that the gradient is one whose components are partial derivatives and is represented as ∇f.

Mp4IMT: IMT stated that the partial derivative of a function of several variables is the derivative of a function with respect to one of its variables and the others are kept constant.

Mp5IMT: IMT found the partial derivative of f with respect to x obtaining $\frac{\partial(x^2y^3-4y)}{\partial x} = 2xy^3$.

Mp6IMT: IMT evaluated the partial derivative with respect to x at the point $P = (2, 1)$, as follows: $\frac{\partial f(2,1)}{\partial x} = 2(2)(1)^3 = 4$, where he used multiplication and exponentiation.

Mp7IMT: IMT found the partial derivative with respect to y, obtaining $\frac{\partial(x^2y^3-4y)}{\partial y} = 3x^2y^2 - 4$.

Mp8IMT: IMT found the partial derivative with respect to y, obtaining: $\frac{\partial f(2,1)}{\partial y} = 3(2)^2(1)^2 - 4 = 8$ (see excerpt from the video transcript of the class).

IMT: Three times two to the power of two, times one to the power of two, minus 4, and how much does that give me? Two times two four, four times three twelve, twelve times one twelve, twelve minus four equals 8.

Mp9IMT: IMT wrote the symbolic expression for the gradient of f equal to: $\nabla f(x,y) = (2xy^3, 3x^2y^2 - 4)$, referring to the partial derivative with respect to x (for the first component) and the partial derivative with respect to y (for the second component), relating it to vector nabla.

Mp10IMT: IMT wrote the symbolic expression of the gradient f evaluated at $P = (2, 1)$ equal to: $\nabla f(2, 1) = (4, 8)$.

Mp11IMT: IMT made a connection of equivalent representations of vector representation $\vec{w} = 2i + 5j$ to rectangular coordinate representation $\vec{w} = (2, 5)$.

Mp12IMT: IMT stated that it must look for a unit vector and, to do so, it must apply the formula $\vec{U} = \frac{\vec{w}}{\|\vec{w}\|}$ and first found the norm of $\vec{w}$ by substituting the components of the vector in the formula $\|\vec{w}\| = \sqrt{(2)^2 + (5)^2}$.

Mp13IMT: IMT solved the powers of the radicand $\|\vec{w}\| = \sqrt{4 + 25}$.

Mp14IMT: IMT He added the amounts of the radicand and stated that the norm of the vector w is: $\|\vec{w}\| = \sqrt{29}$.

Mp15IMT: IMT structured the unit vector in its vector representation: $\vec{U} = \frac{2}{\sqrt{29}}i + \frac{5}{\sqrt{29}}j$.

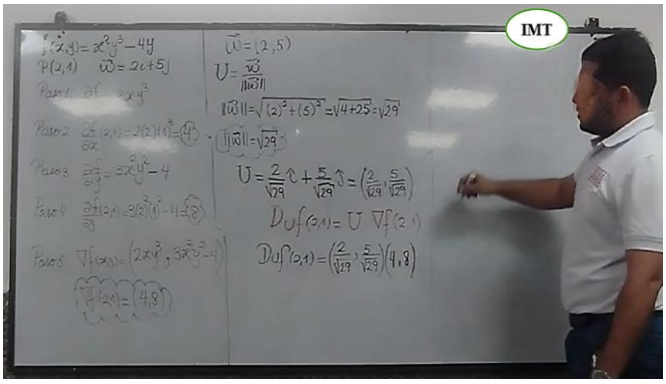

Figure 9. IMT by applying the dot product (Source: Authors' own elaboration)

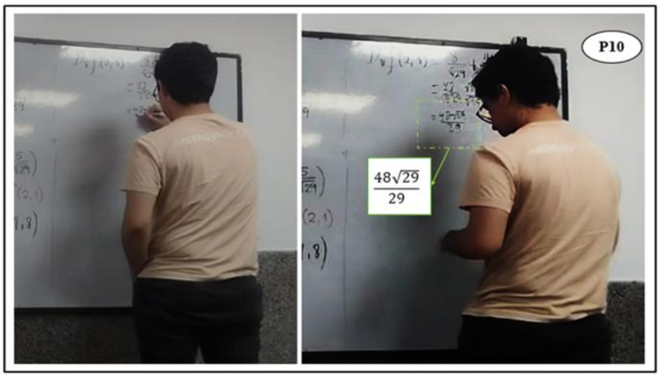

Figure 10. P10 participated in class and applied rationalization (Source: Authors' own elaboration)

Mp16IMT: IMT expressed the unit vector in an equivalent way $\vec{U} = \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$.

Mp17IMT: IMT stated that it has obtained the unit vector, and the gradient evaluated at $P = (2, 1)$ which are requirements to find the directional derivative with the formula $D_U f(2, 1) = U \cdot \nabla f(2, 1)$.

Mp18IMT: IMT replaced the unit vector and gradient in the formula $D_U f(2, 1) = (4,8) \cdot \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$, see **Figure 9**.

Mp19IMT: IMT applied the dot product between the two vectors multiplying component by component obtaining: $D_U f(2, 1) = \frac{8}{\sqrt{29}} + \frac{40}{\sqrt{29}}$.

Mp20IMT: IMT added the fractions to obtain the directional derivative $D_U f(2, 1) = \frac{48}{\sqrt{29}}$.

After IMT obtained the directional derivative, a student (P10) participated in the class and said that $\frac{48}{\sqrt{29}}$ can be rationalized. So, P10 came to the board, rationalized, and got that the directional derivative is $\frac{48\sqrt{29}}{29}$, see **Figure 10**.

P59 mathematical practices (MpP59)

Mp1P59: P59 read and understood task 1 by identifying the function, the direction of the vector U and the point $P = (0, 3)$.

Mp2P59: P59 stated that the directional derivative is the rate of change of the function in the direction of a vector.

Mp3P59: P59 considered the formula $D_U f(x, y) = U \cdot \nabla f(x, y)$ referring to the fact that the directional derivative is found by means of the dot product between the unit vector and the gradient of the function.

Mp4P59: P59 stated that the gradient is a vector whose components are partial derivatives.

Mp5P59: P59 stated that, $D_U f(x, y) = U \cdot \nabla f(x, y)$ is equivalent to $D_U f(x, y) = (cos(\theta), sin(\theta)) \cdot (f_x(x, y), f_y(x, y))$.

Mp6P59: P59 developed the dot product expressed in the previous formula obtaining the expression $D_U f(x, y) = \frac{\partial f(x,y)}{\partial x} cos\,(\theta) + \frac{\partial f(x,y)}{\partial y} sin\,(\theta)$ to find the directional derivative.

Mp7P59: P59 substituted the function and the angle and substituted them into the formula expressed, as follows: $D_U f(x, y) = \frac{\partial}{\partial x}(2x^2 + 5y^2 + 5x^3y - 8y)cos\left(\frac{\pi}{4}\right) + \frac{\partial}{\partial y}(2x^2 + 5y^2 + 5x^3y - 8y)sin\left(\frac{\pi}{4}\right)$.

Mp8P59: P59 found the partial derivative of the function f with respect to x and obtained $4x + 15x^2y$.

Mp9P59: P59 found the partial derivative of the function f with respect to y, obtaining: $10y + 5x^3 - 8$.

Mp10P59: P59 replaced the partial derivatives in the formula $D_U f(x, y) = (4x + 15x^2y)cos\left(\frac{\pi}{4}\right) + (10y + 5x^3 - 8)sin\left(\frac{\pi}{4}\right)$.

Mp11P59: P59 determined the values of $sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

Mp12P59: P59 replaced the sine and cosine values in the formula: $\boldsymbol{D_U f(x, y)} = (4x + 15x^2y)\left(\frac{\sqrt{2}}{2}\right) + (10y + 5x^3 - 8)\left(\frac{\sqrt{2}}{2}\right)$**.**

Mp13P59: P59 applied the distributive property$D_U f(x, y) = \left[(4x)\left(\frac{\sqrt{2}}{2}\right) + (15x^2y)\left(\frac{\sqrt{2}}{2}\right)\right] +$

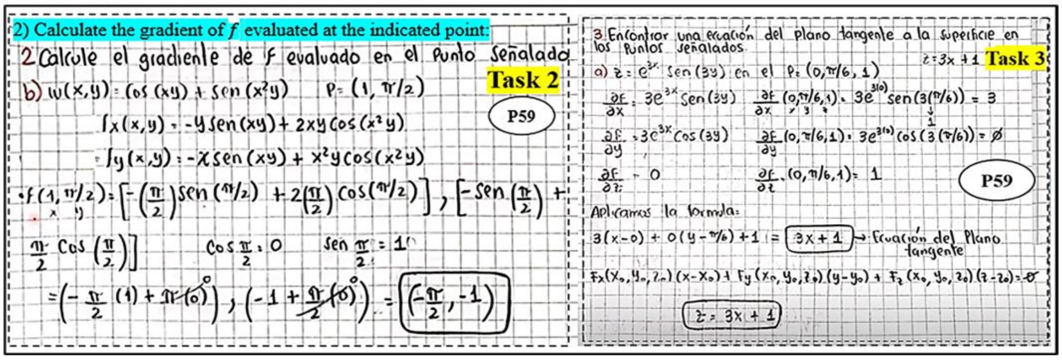

Figure 11. Resolution of task 2 and task 3 by P59 (Source: Authors' own elaboration)

$\left[(10y)\left(\frac{\sqrt{2}}{2}\right)+(5x^3)\left(\frac{\sqrt{2}}{2}\right)-8\left(\frac{\sqrt{2}}{2}\right)\right]$ considering arithmetic operations.

Mp14P59: P59 simplified the directional derivative by obtaining: $D_U f(x,y)=\left[(2x)(\sqrt{2})+\frac{(15x^2y\sqrt{2})}{2}\right]+\left[(5y)(\sqrt{2})+\frac{(5x^3)(\sqrt{2})}{2}-(4\sqrt{2})\right]$.

Mp15P59: P59 replaced the point $P=(0,3)$ in the directional derivative $D_U f(0,3)=\left[(2(0))(\sqrt{2})+\frac{(15(0)^2(3)\sqrt{2})}{2}\right]+\left[(5(3))(\sqrt{2})+\frac{(5(0)^3)(\sqrt{2})}{2}-(4\sqrt{2})\right]$.

Mp16P59: P59 applied arithmetic operations to obtain the directional derivative at the point $P=(0,3)$: $D_U f(0,3)=15\sqrt{2}-4\sqrt{2}=11\sqrt{2}$.

Similarly, the mathematical practices of the other four tasks proposed in the questionnaire are analyzed as presented in **Figure 11** (task 2 and task 3), with examples of the resolution of P59.

Figure 12 shows P59's resolution of task 4.

Other students such as P73, P76, P77, P124, P125, P133, P136, P158, P160, P161, among others, performed tasks 3, 4, and 5 correctly, obtaining the same answer as P59, but with different steps.

In relation to the resolution of task 5, P59 considered the formula $div\,\vec{F}(x,y,z)=\nabla\cdot\vec{F}$ and then the nabla vector and the components of the vector function in the above formula. Later, they found the following partial derivatives: $\frac{\partial(2x)}{\partial x}=2$; $\frac{\partial(3y^2)}{\partial y}=6y$; $\frac{\partial(-5z^3)}{\partial z}=-15z^2$. Then, P59 added the results of the partial derivatives because it is a dot product and obtained: $div\,\vec{F}(x,y,z)=2+6y-15z^2$. Finally, they evaluated the divergence in $P=(1,2,-1)$ to obtain the divergence at a point:

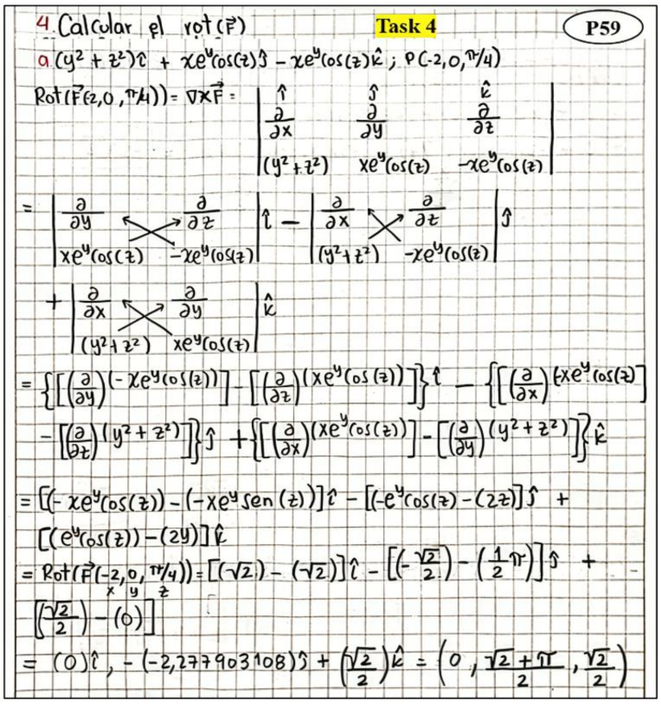

Figure 12. Evidence of resolution of task 4 by P59 (Source: Authors' own elaboration)

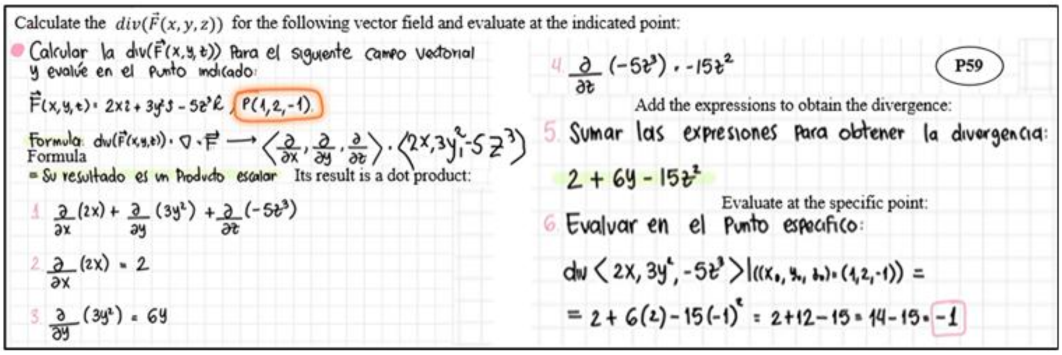

Figure 13. Written production of P59 when solving task 5 (Source: Authors' own elaboration)

$div\ \vec{F}(1,2,-1) = 2 + 6(2) - 15(-1)^2 = 14 - 15 = -1$, see **Figure 13**.

Cognitive Configurations

Several students participated in this research, but the cognitive configurations of primary objects that will be presented. We will only be exemplified with the mathematical activity of the teacher (IMT) and P59 and other cases that proceeded in a similar way when solving some tasks.

IMT cognitive configuration

This section presents the cognitive configuration of the IMT that brings together the primary objects that emerged from its mathematical activity to find the directional derivative (**Table 3**).

Table 3. IMT's cognitive configuration

PO	Description
Problem situation (task)	Determine the directional derivative of the function given by $f(x,y) = x^2y^3 - 4y$ at the point $P = (2,1)$ in the direction of the vector $w = 2i + 5j$.
Language	*Verbal:* Point, function, line, plane, derivative, partial derivative, partial derivative at a point, directional derivative, vector, gradient, unit vector, norm of vector, dot product, among others. *Symbolic:* $f(x,y) = x^2y^3 - 4y$; $P = (2,1)$; $\vec{w} = 2i + 5j$; $\frac{\partial(x^2y^3-4y)}{\partial x} = 2xy^3$; $\frac{\partial f(2,1)}{\partial x} = 2(2)(1)^3 = 4$; $\frac{\partial(x^2y^3-4y)}{\partial y} = 3x^2y^2 - 4$; $\frac{\partial(2,1)}{\partial y} = 3(2)^2(1)^2 - 4 = 8$; $\nabla f(x,\ y) = (2xy^3, 3x^2y^2 - 4)$; $\nabla f(2,\ 1) = (4,8)$; $\vec{w} = 2i + 5j$; $\vec{w} = (2,5)$; $\vec{U} = \frac{\vec{w}}{\lVert\vec{w}\rVert}$; $\vec{w} = \sqrt{(2)^2 + (5)^2}$; $\vec{w} = \sqrt{4+25} = \sqrt{29}$; $\vec{w} = \sqrt{29}$; $\vec{U} = \frac{2}{\sqrt{29}}i + \frac{5}{\sqrt{29}}j$; $\vec{U} = \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$; $D_Uf(2,\ 1) = U \cdot \nabla f(2,\ 1)$; $D_Uf(2,\ 1) = (4,8) \cdot \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$; $D_Uf(2,\ 1) = \frac{8}{\sqrt{29}} + \frac{40}{\sqrt{29}}$; $D_Uf(2,\ 1) = \frac{48}{\sqrt{29}}$; $\frac{48\sqrt{29}}{29}$.
Concepts/ definitions	*Previous concepts*: Point, function, line, plane, derivative, partial derivative, partial derivative at a point, directional derivative, vector, gradient, unit vector, norm of a vector, dot product, among others. Definitions (D): D1: The directional derivative is understood as the rate of change of the function in the direction of a vector. D2: Vector nabla or gradient is one whose components are partial derivatives and is denoted by ∇f. D3: The partial derivative of a function of several variables is the derivative of the function with respect to each of those variables, whether x, y, z, etc. D4: A vector is a line segment with magnitude, sense, and direction. D5: Unit vector is a vector with a magnitude equal to 1.
Propositions/ properties	*Previous propositions:* Arithmetic operations, determinants, etc. Proposition 1 (Pr1): The partial derivative of f with respect to x $\left[\frac{\partial(x^2y^3-4y)}{\partial x} = 2xy^3\right]$ evaluated at the point $P = (2,1)$ is: $\frac{\partial f(2,1)}{\partial x} = 2(2)(1)^3 = 4$. Pr2: The partial derivative with respect to y is $\left[\frac{\partial(x^2y^3-4y)}{\partial y} = 3x^2y^2 - 4\right]$ evaluated at the point $P = (2,1)$ is: $\frac{\partial f(2,1)}{\partial y} = 3(2)^2(1)^2 - 4 = 8$. Pr3: The gradient of f $[\nabla f(x,\ y) = (2xy^3, 3x^2y^2 - 4)]$, evaluated in $P = (2,1)$ is equal to: $\nabla f(2,\ 1) = (4,8)$. Pr4: The norm of the vector w is: $\vec{w} = \sqrt{29}$. Pr5: The unit vector is: $\vec{U} = \frac{2}{\sqrt{29}}i + \frac{5}{\sqrt{29}}j$, equivalent to: $\vec{U} = \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$. Pr6: The directional derivative $D_Uf(2,\ 1) = \frac{48}{\sqrt{29}}$ equivalent to: $\frac{48\sqrt{29}}{29}$.
Procedures	*Great procedure (GP):* Find the directional derivative of the function $f(x,y) = x^2y^3 - 4y$ at $P = (0,1)$ in the direction $\vec{w} = 2i + 5j$. Main procedure 1 (MPc1): Find the gradient of the function evaluated at $P = (2,1)$. Auxiliary procedure 1.1 (APc1.1): He found the partial derivative with respect to x of $f(x,y) = 2xy^3$. APc1.2: IMT evaluated the partial derivative with respect to x at $P = (2,1)$ iqual to 4. APc1.3: IMT found the partial derivative with respect to y of $f(x,y) = x^2y^3 - 4y$. APc1.4: IMT evaluated the partial derivative with respect to y in $P = (2,1)$ iqual to 8. APc1.5: IMT symbolically represented the gradient of the function: $[\nabla f(x,\ y) = (2xy^3, 3x^2y^2 - 4)]$. APc1.6: IMT symbolically represented the gradient of the function evaluated in $P = (2,1)$: $\nabla f(2,\ 1) = (4,8)$. MPc2: Find the unit vector of $\vec{w} = 2i + 5j$. APc2.1: IMT used equivalent representations $\vec{w} = 2i + 5j$ is equal to $\vec{w} = (2,5)$. APc2.2: IMT applied the formula $\vec{U} = \frac{\vec{w}}{\lVert\vec{w}\rVert}$ and first found the norm of $\vec{w}$: $\lVert\vec{w}\rVert = \sqrt{(2)^2 + (5)^2}$. APc2.3: IMT solved the powers in the radicand $\lVert\vec{w}\rVert = \sqrt{4+25}$ APc2.4: IMT added the radicand amounts and found the norm of the vector: $\lVert\vec{w}\rVert = \sqrt{29}$. APc2.5: IMT obtained the unit vector: $\vec{U} = \frac{2}{\sqrt{29}}i + \frac{5}{\sqrt{29}}j$, and said it is equivalent to: $\vec{U} = \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$. MPc3: Find the directional derivative considering the gradient evaluated at the point: $\nabla f(2,\ 1) = (4,8)$ and the unit vector: $\vec{U} = \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$. APc3.1: IMT applied the formula $D_Uf(x,\ y) = U \cdot \nabla f(x,\ y)$ evaluated in $P = (0,1)$: $D_Uf(2,\ 1) = U \cdot \nabla f(2,\ 1)$.

Table 3 (Continued).

PO	Description
	APc3.4: IMT added the fractions and got the directional derivative: $D_U f(2,\ 1) = \frac{48}{\sqrt{29}}$ equivalent to $\frac{48\sqrt{29}}{29}$. APc3.3: IMT applied the dot product multiplying component by component of each vector and obtained: $D_U f(2,\ 1) = \frac{8}{\sqrt{29}} + \frac{40}{\sqrt{29}}$. APc3.2: IMT replaced the unit vector and the gradient: $D_U f(2,\ 1) = (4{,}8) \cdot \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$.
Arguments	**Argument 1 (A1):** *Thesis:* The partial derivative of f with respect to x is $\left[\frac{\partial(x^2y^3-4y)}{\partial x} = 2xy^3\right]$ evaluated at the point $P = (2{,}1)$ is: $\frac{\partial f(2{,}1)}{\partial x} = 2(2)(1)^3 = 4$. Reason 1 (R1): IMT derived the function with respect to x using the derivative of a power function. R2: IMT evaluates the derivative with respect to x at the point $P = (2{,}1)$ to obtain the partial derivative 4. Conclusion (C): Actually the partial derivative of f with respect to x is 4.
	A2: *Thesis:* The partial derivative of f with respect to y is $\left[\frac{\partial(x^2y^3-4y)}{\partial x} = 3x^2y^2 - 4\right]$ evaluated at the point $P = (2{,}1)$ is: $\frac{\partial f(2{,}1)}{\partial y} = 3(2)^2(1)^2 - 4 = 8$. R1: IMT derived the function with respect to y using the derivative of a power function. R2: IMT evaluates the derivative with respect to y at the point $P = (2{,}1)$ to obtain the partial derivative 8. Conclusion: Actually the partial derivative of f with respect to y is 8
	A3: *Thesis:* The gradient of evaluated at $P = (2{,}1)$ is equal to $\nabla f(2,\ 1) = (4, 8)$. R1: IMT constructs and symbolizes the gradient with the partial derivative components with respect to x and y: $\nabla f(x,\ y) = (2xy^3, 3x^2y^2 - 4)$ R2: IMT evaluates the gradient at $P = (2{,}1)$ to obtain $\nabla f(2,\ 1) = (4, 8)$. Conclusion: Based on the partial derivatives it was found that the gradient is truly $(4, 8)$.
	A4: *Thesis:* The norm of the vector $\boldsymbol{w}$ is: $\lVert\vec{\boldsymbol{w}}\rVert = \sqrt{\mathbf{29}}$. R1: IMT represents equivalent vectors: $\vec{\boldsymbol{w}} = \mathbf{2}\boldsymbol{i} + \mathbf{5}\boldsymbol{j}$ **is equal** to $\vec{\boldsymbol{w}} = (\mathbf{2}, \mathbf{5})$. R2: IMT applies the norm formula $\lVert\vec{\boldsymbol{w}}\rVert = \sqrt{(\mathbf{2})^2 + (\mathbf{5})^2}$ R3: Perform operations with potentiation, multiplications and additions to obtain: $\lVert\vec{\boldsymbol{w}}\rVert = \sqrt{\mathbf{4} + \mathbf{25}} = \sqrt{\mathbf{29}}$. Conclusion: Actually the norm of the vector $\boldsymbol{w}$ is: $\lVert\vec{\boldsymbol{w}}\rVert = \sqrt{\mathbf{29}}$.
	A5: *Thesis:* The unit vector is $\vec{\boldsymbol{U}} = \frac{\mathbf{2}}{\sqrt{\mathbf{29}}}\boldsymbol{i} + \frac{\mathbf{5}}{\sqrt{\mathbf{29}}}\boldsymbol{j} = \left(\frac{\mathbf{2}}{\sqrt{\mathbf{29}}}, \frac{\mathbf{5}}{\sqrt{\mathbf{29}}}\right)$ R1: IMT followed the unit vector formula: $\vec{\boldsymbol{U}} = \frac{\vec{\boldsymbol{w}}}{\lVert\vec{\boldsymbol{w}}\rVert}$ to structure the vector $\vec{\boldsymbol{U}} = \frac{\mathbf{2}}{\sqrt{\mathbf{29}}}\boldsymbol{i} + \frac{\mathbf{5}}{\sqrt{\mathbf{29}}}\boldsymbol{j}$ in its vector representation. R2: IMT equivalently represents the unit vector $\left(\frac{\mathbf{2}}{\sqrt{\mathbf{29}}}, \frac{\mathbf{5}}{\sqrt{\mathbf{29}}}\right)$. Conclusion: Finally the unit vector is $\left(\frac{\mathbf{2}}{\sqrt{\mathbf{29}}}, \frac{\mathbf{5}}{\sqrt{\mathbf{29}}}\right)$.
	A6: *Thesis:* The directional derivative is $D_U f(2,\ 1) = \frac{48}{\sqrt{29}}$ equivalent to: $\frac{48\sqrt{29}}{29}$. R1: IMT applies to the dot product between the unit vector and the gradient vector. R2: IMT performs addition of fractions and obtains the directional derivative $D_U f(2,\ 1) = \frac{48}{\sqrt{29}}$. R3: IMT with the help of a student rationalizes the expression $\frac{48}{\sqrt{29}}$ getting an equivalent expression $\frac{48\sqrt{29}}{29}$. Conclusion: The directional derivative is $\frac{48\sqrt{29}}{29}$.

P59 cognitive configuration

The cognitive configuration of P59 is constructed for task 1 considering the temporal narrative written before and the mathematical practices given before.

Subsequently, with the information provided in mathematical practices and in the cognitive configurations of IMT and P59, the SFs that will be presented in section 4.5 are established (**Table 4**).

Table 4. P59's cognitive configuration

PO	Description
Problem situation (task)	Calculate the directional derivative for the function $f(x, y) = 2x^2 + 5y^2 + 5x^3 - 8y$ in the direction $U = cos\left(\frac{\pi}{4}\right)i + sin\left(\frac{\pi}{4}\right)j$ and evaluate said derivative at the indicated point $D_u f(0{,}3)$.
Language	*Verbal*: Point, function, line, plane, derivative, partial derivative, partial derivative at a point, directional derivative, vector, gradient, unit vector, norm of vector, dot product, among others..
	Symbolic: $D_U f(x,\ y) = U \cdot \nabla f(x,\ y)$, $\boldsymbol{D_U f(x,\ y)} = \left(\boldsymbol{cos(\theta)}, \boldsymbol{sin(\theta)}\right) \cdot \left(\boldsymbol{f_x(x,\ y)}, \boldsymbol{f_y(x,\ y)}\right)$, $\boldsymbol{D_U f(x, y)} = \frac{\boldsymbol{\partial f(x,y)}}{\boldsymbol{\partial x}}\boldsymbol{cos\ (\theta)} + \frac{\boldsymbol{\partial f(x,y)}}{\boldsymbol{\partial y}}\boldsymbol{sin\ (\theta)}$, $\boldsymbol{D_U f(x, y)} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}}(\mathbf{2}\boldsymbol{x}^\mathbf{2} + \mathbf{5}\boldsymbol{y}^\mathbf{2} + \mathbf{5}\boldsymbol{x}^\mathbf{3}\boldsymbol{y} - \mathbf{8}\boldsymbol{y})\boldsymbol{cos}\left(\frac{\boldsymbol{\pi}}{\mathbf{4}}\right) + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial y}}(\mathbf{2}\boldsymbol{x}^\mathbf{2} + \mathbf{5}\boldsymbol{y}^\mathbf{2} + \mathbf{5}\boldsymbol{x}^\mathbf{3}\boldsymbol{y} - \mathbf{8}\boldsymbol{y})\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\mathbf{4}}\right)$, $4x + 15x^2y$, $10y + 5x^3 - 8$, $sin\ \left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $\boldsymbol{D_U f(x, y)} = (4x + 15x^2y)\left(\frac{\sqrt{2}}{2}\right) + (10y + 5x^3 - 8)\left(\frac{\sqrt{2}}{2}\right)$, $\boldsymbol{D_U f(x, y)} = \left[(\mathbf{4}\boldsymbol{x})\left(\frac{\sqrt{\mathbf{2}}}{\mathbf{2}}\right) + (\mathbf{15}\boldsymbol{x}^\mathbf{2}\boldsymbol{y})\left(\frac{\sqrt{\mathbf{2}}}{\mathbf{2}}\right)\right] + \left[(\mathbf{10}\boldsymbol{y})\left(\frac{\sqrt{\mathbf{2}}}{\mathbf{2}}\right) + (\mathbf{5}\boldsymbol{x}^\mathbf{3})\left(\frac{\sqrt{\mathbf{2}}}{\mathbf{2}}\right) - \mathbf{8}\left(\frac{\sqrt{\mathbf{2}}}{\mathbf{2}}\right)\right]$, $\boldsymbol{D_U f(x, y)} =$

Table 4 (Continued). P59's cognitive configuration

PO	Description
	$\left[(\mathbf{2x})(\sqrt{\mathbf{2}}) + \frac{(15x^2y\sqrt{2})}{2}\right] + \left[(\mathbf{5y})(\sqrt{\mathbf{2}}) + \frac{(5x^3)(\sqrt{2})}{2} - (\mathbf{4}\sqrt{\mathbf{2}})\right]$, $P = (0,3)$, $D_Uf(0,3) = \left[(2(0))(\sqrt{2}) + \frac{(15(0)^2(3)\sqrt{2})}{2}\right] + \left[(5(3))(\sqrt{2}) + \frac{(5(0)^3)(\sqrt{2})}{2} - (4\sqrt{2})\right]$, $D_Uf(0,3) = 15\sqrt{2} - 4\sqrt{2} = 11\sqrt{2}$.
Concepts/ definitions	*Previous concepts*: Point, function, line, plane, derivative, partial derivative, partial derivative at a point, directional derivative, vector, gradient, unit vector, norm of a vector, dot product, among others. Definitions (D): D1: The directional derivative is defined as the rate of change of the function in the direction of a vector. D2: Nabla vector or gradient is the one whose components are partial derivatives and is denoted by ∇f. D3: The partial derivative of a function of several variables is the derivative of the function with respect to each one of those variables, either x, y, z, etc. D4: A vector is a line segment with magnitude, sense, and direction. D5: Unit vector is a vector with a magnitude equal to 1.
Propositions/ properties	*Previous propositions*: Arithmetic operations, determinants, distributive property, etc. Proposition 1 (Pr1): P59 stated that the directional derivative is found with the formula $D_Uf(x,\ y) = U \cdot \nabla f(x,\ y)$. Pr2: P59 stated that with the formula $D_Uf(x,y) = \frac{\partial f(x,y)}{\partial x} cos\ (\theta) + \frac{\partial f(x,y)}{\partial y} sin\ (\theta)$ the directional derivative is found by specifying the dot product, unit vector and partial derivatives. Pr3: P59 said that partial derivative with respect to x is $4x + 15x^2y$. Pr4: P59 said that partial derivative with respect to y is $10y + 5x^3 - 8$. Pr5: P59 mentioned that the general directional derivative is $D_Uf(x,y) = \left[(2x)(\sqrt{2}) + \frac{(15x^2y\sqrt{2})}{2}\right] + \left[(5y)(\sqrt{2}) + \frac{(5x^3)(\sqrt{2})}{2} - (4\sqrt{2})\right]$. Pr6: P59 found *that* the directional derivative at point $P = (0,3)$ is $D_Uf(0,3) = 15\sqrt{2} - 4\sqrt{2} = 11\sqrt{2}$.
Procedures	Great procedure (GP): Calculate the directional derivative for the function $f(x,y) = 2x^2 + 5y^2 + 5x^3 - 8y$ in the direction $U = cos\left(\frac{\pi}{4}\right) i + sin\left(\frac{\pi}{4}\right) j$ evaluated in $P = (0,3)$. Main procedure 1 (MPc1): explicitly obtain the formula to find the directional derivative. Auxiliary procedure 1.1 (APc1.1): P59 expressed the formula of the directional derivative $D_Uf(x,\ y) = U \cdot \nabla f(x,\ y)$. APc1.2: P59 replaced the unit vector in its trigonometric form and the gradient with the components as partial derivatives of the function: $D_Uf(x,\ y) = (cos(\theta), sin(\theta)) \cdot \left(f_x(x,\ y), f_y(x,\ y)\right)$. APc1.3: P59 executed the dot product between the unit vector and the gradient, multiplying component by component: $D_Uf(x,y) = \frac{\partial f(x,y)}{\partial x} cos\ (\theta) + \frac{\partial f(x,y)}{\partial y} sin\ (\theta)$. Hereafter, it is recognized that P59 follows the procedures using the formula obtained in APc1.3. However, it will now be shown how P59 obtained the gradient and then the unit vector and substitutes them into the formula. MPc2: obtain the directional derivative of the function in general form. APc2.1: P59 substituted the function and the unit vector in the formula: $D_Uf(x,y) = \frac{\partial}{\partial x}(2x^2 + 5y^2 + 5x^3y - 8y) cos\left(\frac{\pi}{4}\right) + \frac{\partial}{\partial y}(2x^2 + 5y^2 + 5x^3y - 8y) sin\left(\frac{\pi}{4}\right)$. APc2.2: P59 found the partial derivative with respect to x: $4x + 15x^2y$. APc2.3: P59 found the partial derivative with respect to y: $10y + 5x^3 - 8$. APc2.4: P59 substituted the partial derivatives obtained in APc2.2 and APc2.3 in the formula: $\boldsymbol{D_Uf(x,y)} = (4x + 15x^2y) cos\left(\frac{\pi}{4}\right) + (10y + 5x^3 - 8) sin\ \left(\frac{\pi}{4}\right)$. APc2.5: P59 obtained the values of the $sin\ \left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ fundamental to find the components of the unit vector. APc2.6: P59 substituted the components of the unit vector in the formula: $\boldsymbol{D_Uf(x,y)} = (4x + 15x^2y)\left(\frac{\sqrt{2}}{2}\right) + (10y + 5x^3 - 8)\left(\frac{\sqrt{2}}{2}\right)$. APc2.7: P59 applied the distributive property to the expression obtained in APc2.5, resulting: $\boldsymbol{D_Uf(x,y)} = \left[(\mathbf{4x})\left(\frac{\sqrt{2}}{2}\right) + (\mathbf{15x^2y})\left(\frac{\sqrt{2}}{2}\right)\right] + \left[(\mathbf{10y})\left(\frac{\sqrt{2}}{2}\right) + (\mathbf{5x^3})\left(\frac{\sqrt{2}}{2}\right) - \mathbf{8}\left(\frac{\sqrt{2}}{2}\right)\right]$ APc2.8: P59 uses simplification to reduce the expression: $\boldsymbol{D_Uf(x,y)} = \left[(\mathbf{2x})(\sqrt{\mathbf{2}}) + \frac{(15x^2y\sqrt{2})}{2}\right] + \left[(\mathbf{5y})(\sqrt{\mathbf{2}}) + \frac{(5x^3)(\sqrt{2})}{2} - (\mathbf{4}\sqrt{\mathbf{2}})\right]$ MPc3: Obtain the directional derivative at the point $P = (0,3)$. APc3.1: P59 substituted $P = (0,3)$ in the global directional derivative and obtained: $D_Uf(0,3) = \left[(2(0))(\sqrt{2}) + \frac{(15(0)^2(3)\sqrt{2})}{2}\right] + \left[(5(3))(\sqrt{2}) + \frac{(5(0)^3)(\sqrt{2})}{2} - (4\sqrt{2})\right]$. APc3.2. After doing arithmetic operations and radicals, P59 got the derivative at a point $D_Uf(0,3) = 15\sqrt{2} - 4\sqrt{2} = 11\sqrt{2}$ (**Figure 14**).

Table 4 (Continued). P59's cognitive configuration

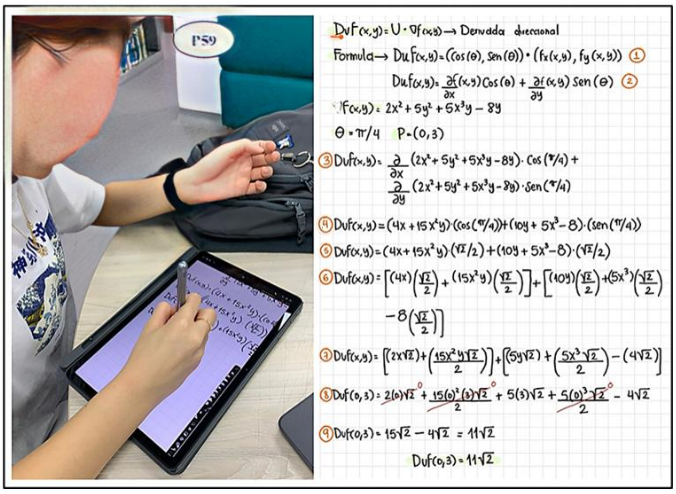

Figure 14. Procedure performed by P59 (Source: Authors' own elaboration)

PO	Description
Arguments	**A1:** *Thesis:* The formula $D_U f(x,\ y) = U \cdot \nabla f(x,\ y)$ is useful for finding the directional derivative. R1: P59 mentions that the directional derivative is obtained by performing the dot product between the nabla vector and the unit vector. Conclusion: Indeed, the formula is useful to find the directional derivative **A2:** *Thesis:* The explicit formula for finding directional derivative is $D_U f(x,y) = \frac{\partial f(x,y)}{\partial x} cos\ (\theta) + \frac{\partial f(x,y)}{\partial y} sin\ (\theta)$. R1: P59 executed the dot product between $D_U f(x,\ y) = \left(cos(\theta), sen(\theta)\right) \cdot \left(f_x(x,\ y), f_y(x,\ y)\right)$. Conclusion: After running the dot product the proper formula for finding the directional derivative is: $D_U f(x,y) = \frac{\partial f(x,y)}{\partial x} cos\ (\theta) + \frac{\partial f(x,y)}{\partial y} sen\ (\theta)$. **A3:** *Thesis:* The partial derivative with respect to x is $4x + 15x^2y$. R1: P59 used the partial derivative rule with respect to x of the power function $\frac{\partial}{\partial x}(2x^2 + 5y^2 + 5x^3y - 8y)$. Conclusion: Actually, the partial derivative of f with respect to x is $4x + 15x^2y$. **A4:** *Thesis:* The partial derivative with respect to y is $10y + 5x^3 - 8$. R1: P59 used the rule of partial derivative with respect to y of the power function $\frac{\partial}{\partial y}(2x^2 + 5y^2 + 5x^3y - 8y)$. Conclusion: Actually, the partial derivative of f with respect to y is $10y + 5x^3 - 8$. **A5:** *Thesis:* The general directional derivative is $D_U f(x,y) = \left[(2x)(\sqrt{2}) + \frac{(15x^2y\sqrt{2})}{2}\right] + \left[(5y)(\sqrt{2}) + \frac{(5x^3)(\sqrt{2})}{2} - (4\sqrt{2})\right]$. R1: P59 substituted the partial derivatives obtained in APc2.2 and APc2.3 in the formula: $\boldsymbol{D_U f(x,y)} = (4x + 15x^2y)cos\left(\frac{\pi}{4}\right) + (10y + 5x^3 - 8)sin\left(\frac{\pi}{4}\right)$. R2: P59 obtained the values of the $sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. R3: P59 substituted the components of the unit vector in the formula: $\boldsymbol{D_U f(x,y)} = (4x + 15x^2y)\left(\frac{\sqrt{2}}{2}\right) + (10y + 5x^3 - 8)\left(\frac{\sqrt{2}}{2}\right)$. R4: P59 applied the distributive property to the written expression in R3 resulting in: $\boldsymbol{D_U f(x,y)} = \left[\boldsymbol{(4x)}\left(\frac{\sqrt{2}}{2}\right) + \boldsymbol{(15x^2y)}\left(\frac{\sqrt{2}}{2}\right)\right] + \left[\boldsymbol{(10y)}\left(\frac{\sqrt{2}}{2}\right) + \boldsymbol{(5x^3)}\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{8}\left(\frac{\sqrt{2}}{2}\right)\right]$ R5: P59 simplified the above expression. Conclusion: Effectively the directional derivative in its general form is: $\boldsymbol{D_U f(x,y)} = \left[\boldsymbol{(2x)(\sqrt{2})} + \frac{\boldsymbol{(15x^2y\sqrt{2})}}{\boldsymbol{2}}\right] + \left[\boldsymbol{(5y)(\sqrt{2})} + \frac{\boldsymbol{(5x^3)(\sqrt{2})}}{\boldsymbol{2}} - \boldsymbol{(4\sqrt{2})}\right]$. **A6:** *Thesis:* The directional derivative of f at the point $P = (0,3)$ is: $11\sqrt{2}$ R1: P59 substituted P = (0, 3) in the general directional derivative obtaining: $D_U f(0,3) = \left[(2(0))(\sqrt{2}) + \frac{(15(0)^2(3)\sqrt{2})}{2}\right] + \left[(5(3))(\sqrt{2}) + \frac{(5(0)^3)(\sqrt{2})}{2} - (4\sqrt{2})\right]$. R2: P59 performed arithmetic and radical operations, obtained the directional derivative at a point $D_U f(0,3) = 15\sqrt{2} - 4\sqrt{2} = 11\sqrt{2}$. Conclusion: Finally, the directional derivative at $P = (0,3)$ is $11\sqrt{2}$.

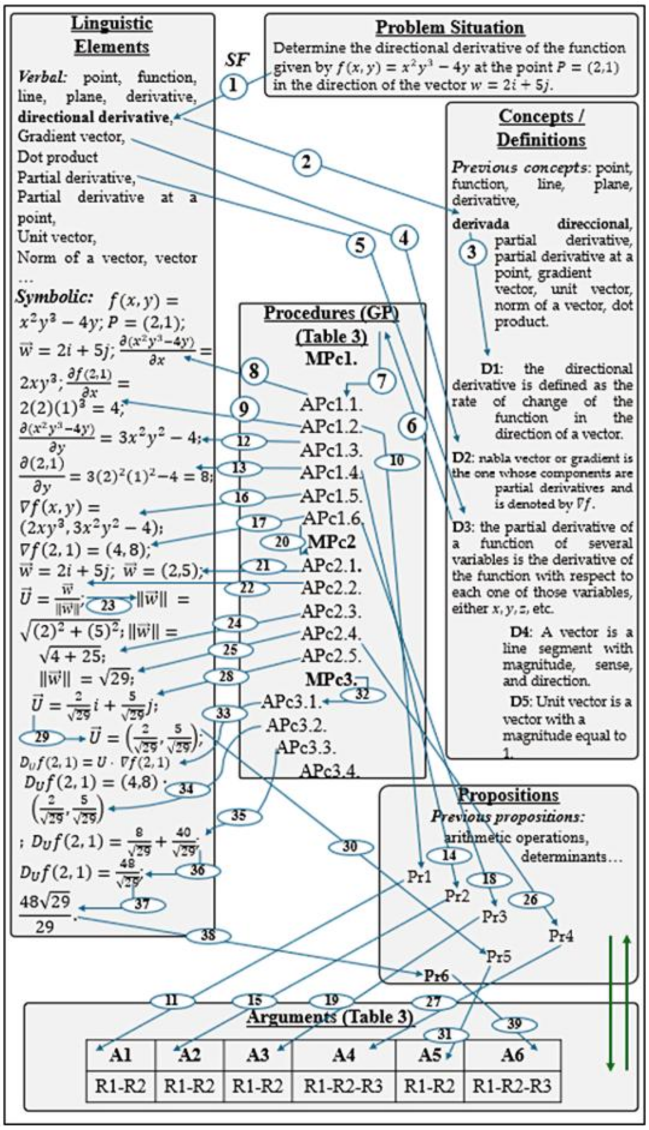

Figure 15. IMT semiotic functions (Source: Authors' own elaboration)

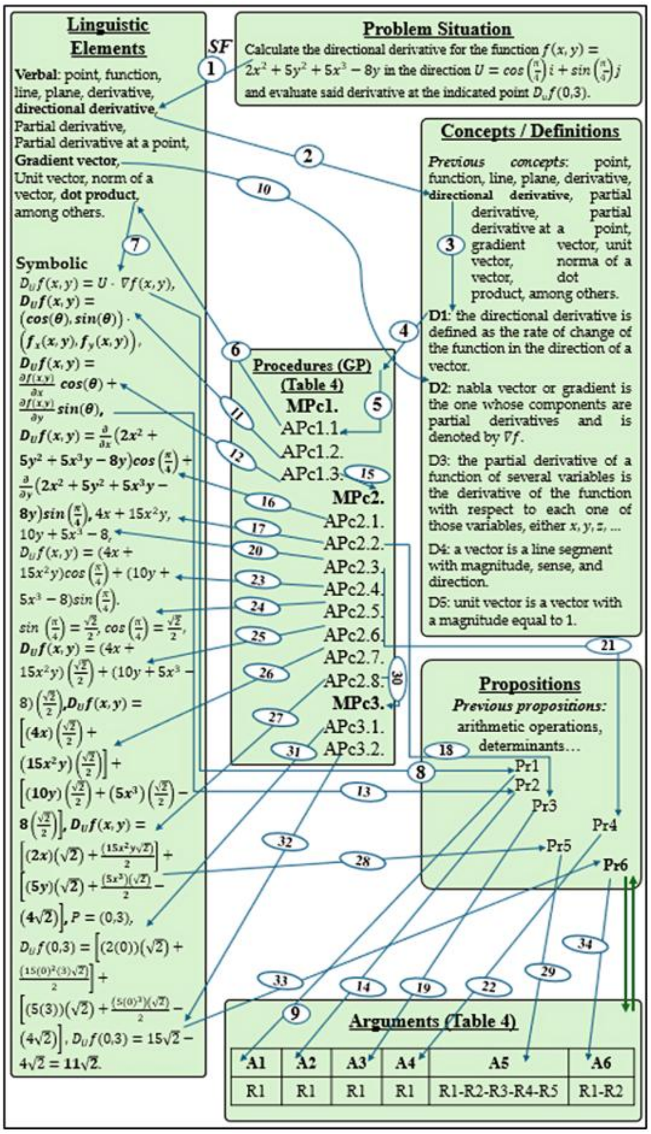

Figure 16. P59 SFs to solve task 1 (Source: Authors' own elaboration)

Semiotic Functions

IMT semiotic functions

The SFs presented in **Figure 15** reflct the sequence of actions performed by the IMT when solving and explaining the problem to its students.

P59 semiotic functions

Figure 15 shows the SFs activated by P59, which highlights the persuasive power of mathematical practices as guides for the sequence of these SFs. In addition, no graphical representations are observed because participants based their resolution on symbolic and/or numerical representations. However, in the answers to the preliminary questions, they used graphical representations related to the tangent plane to the surface. These preliminary graphical representations allowed participants to visualize more concretely the spatial relationships and geometry of the problem, facilitating a more intuitive and deeper understanding of the meanings of vector and partial derivative before addressing the symbolic expressions. This suggests that the integration of different types of representations can enrich the process of solving mathematical problems.

Mathematical Connections

The mathematical connections established by the IMT and P59 are fundamental for solving the tasks proposed in the questionnaire. This detailed analysis makes it possible to visualize the mathematical practices, processes, objects (described in the configurations) and SFs (which relate them) listed sequentially in **Figure 15** and **Figure 16**.

IMT mathematical connections

Table 5 provides relevant information that should be read from left to right (from mathematical practices to mathematical connections), in order to recognize the constitution of each mathematical connection and the vital importance of mathematical practice.

Table 5. Mathematical connections established by IMT to solve task 1

Mp	Processes	Objects	SFs	Mathematical connections
Mp1	Understanding-problematization-enunciation	Derivative, slope, rate of change	SF1	Instruction-oriented
Mp2	Signification/understanding-problem-solving-enunciation	Meaning of the directional derivative	SF2, SF3	Meaning
Mp3	Signification/understanding-problem-solving-enunciation	Meaning of the gradient	SF4	Meaning
Mp4	Signification/understanding-problem-solving-enunciation	Meaning of the partial derivative	SF5	Meaning
Mp5	Problem-solving-algorithmizing-representation	IMT found the partial derivative of f with respect to x, $\frac{\partial(x^2y^3-4y)}{\partial x}=2xy^3$	SF6, SF7, SF8	Procedural
Mp6	Problem-solving-algorithmizing-representation-argumentation	The partial derivative with respect to x at the point $P=(2,1)$ is: $\frac{\partial f(2,1)}{\partial x}=2(2)(1)^3=4$	SF9, SF10, SF11	Procedural
Mp7	Problem-solving-algorithmizing-representation	IMT found the partial derivative of f with respect to y, $\frac{\partial(x^2y^3-4y)}{\partial y}=3x^2y^2-4$	SF12	Procedural
Mp8	Problem-solving-algorithmizing-representation-argumentation	The partial derivative with respect to y at point $P=(2,1)$ is: $\frac{\partial f(2,1)}{\partial y}=3(2)^2(1)^2-4=8$	SF13, SF14, SF15	Procedural
Mp9	Problem-solving-algorithmizing-representation	IMT wrote the symbolic expression of gradient: $\nabla f(x,y)=(2xy^3,3x^2y^2-4)$, referring to the partial derivative with respect to x (for the first component) and the partial derivative with respect to y (for the second component).	SF16	Different representations considering characteristics
Mp10	Problem-solving-algorithmizing representation-argumentation	IMT wrote the symbolic expression of the gradient f evaluated at $P=(2,1)$ iqual to: $\nabla f(2,1)=(4,8)$.	SF17, SF18, SF219	Different representations
Mp11	Problem-solving-representation	IMT transitioned from vector representation $\vec{w}=2i+5j$ to rectangular coordinate representation $\vec{w}=(2,5)$	SF20, SF21	Different representations
Mp12	Problem-solving-algorithmizing	IMT applied the formula $\vec{U}=\frac{\vec{w}}{\lVert\vec{w}\rVert}$ and found the norm of $\vec{w}$: $\lVert\vec{w}\rVert=\sqrt{(2)^2+(5)^2}$.	SF22, SF23	Procedural
Mp13	Problem-solving-algorithmizing-representation	$\lVert\vec{w}\rVert=\sqrt{4+25}$.	SF24	Procedural
Mp14	Problem-solving-argumentation	$\lVert\vec{w}\rVert=\sqrt{29}$.	SF25, SF26, SF27	Procedural
Mp15	Problem-solving-algorithmizing-representation	Unit vector in its vector representation: $\vec{U}=\frac{2}{\sqrt{29}}i+\frac{5}{\sqrt{29}}j$.	SF28	Procedural
Mp16	Problem-solving-representation-argumentation	$\vec{U}=\left(\frac{2}{\sqrt{29}},\frac{5}{\sqrt{29}}\right)$	SF29, SF30, SF31	Different representations
Mp17	Problem-solving-representation	Unit vector and gradient evaluated in $P=(2,1)$, are requirements for finding the directional derivative: $D_Uf(2,1)=U\cdot\nabla f(2,1)$.	SF32, SF33	Instruction-oriented different representations
Mp18	Problem-solving-representation	$D_Uf(2,1)=(4,8)\cdot\left(\frac{2}{\sqrt{29}},\frac{5}{\sqrt{29}}\right)$	SF34	Procedural
Mp19	Problem-solving-algorithmizing-representation	$D_Uf(2,1)=\frac{8}{\sqrt{29}}+\frac{40}{\sqrt{29}}$	SF35	Procedural
Mp20	Problem-solving-algorithmizing-Representation-argumentation	$D_Uf(2,1)=\frac{48}{\sqrt{29}}=\frac{48\sqrt{29}}{29}$	SF36, SF37, SF38, SF39	Procedural

P59 mathematical connections

Based on the information from the mathematical practices of P59 and that is contained in **Figure 16**, the mathematical connections are established in **Table 6** (read from left to right).

Now, to explain in more detail, we exemplify with the procedural mathematical connection associated with the MP8 of the IMT in **Table 5**, using in turn, the definition of mathematical connection (*metaphorically speaking*) as the tip of an iceberg. Other special connections that emerged in the preliminary tasks are those of meaning, different representations and part-whole, where students proceeded adequately in solving the tasks. However, other students presented difficulties in solving the tasks due to mathematical errors caused by the disconnections (personal connections) activated. For

Table 6. P59 mathematical connections

Mp	Processes	Objects	SFs	Mathematical connections
Mp1	Understanding-problematization-enunciation	Identification of the task data and understanding of the problem situation.	SF1	
Mp2	Signification/understanding-problem-solving-enunciation	Meaning of the directional derivative	SF2, SF3	Meaning
Mp3	Signification/understanding-problem-solving-enunciation	Meaning of the directional derivative (represented in the formula and how it can be used)	SF4, SF5, SF6, SF7, SF8, SF9	Meaning-based on different representations
Mp4	Signification/understanding-problem-solving-enunciation	Meaning of the gradient vector	SF10	Meaning
Mp5	Problem-solving-representation	$D_U f(x,\ y) = U \cdot \nabla f(x,\ y)$ is equivalent to $D_U f(x,\ y) = (cos(\theta), sin(\theta)) \cdot (f_x(x,\ y), f_y(x,\ y))$.	SF11	Different representations
Mp6	Problem-solving-algorithmizing-representation-argumentation	Development of the dot product to obtain the explicit formula: $D_U f(x,y) = \frac{\partial f(x,y)}{\partial x} cos\ (\theta) + \frac{\partial f(x,y)}{\partial y} sin\ (\theta)$	SF12, SF13, SF14	Procedural
Mp7	Problem-solving-algorithmizing -representation	$D_U f(x,y) = \frac{\partial}{\partial x}(2x^2 + 5y^2 + 5x^3y - 8y)cos\left(\frac{\pi}{4}\right) + \frac{\partial}{\partial y}(2x^2 + 5y^2 + 5x^3y - 8y)sin\left(\frac{\pi}{4}\right)$.	SF15, SF16	Procedural
Mp8	Problem-solving-algorithmizing-representation-argumentation	Partial derivative of the function f with respect to x y obtained $4x + 15x^2y$.	SF17, SF18, SF19	Procedural
Mp9	Problem-solving-algorithmizing-representation-argumentation	Partial derivative of the function f with respect to y, obtaining: $10y + 5x^3 - 8$	SF20, SF21, SF22	Procedural
Mp10	Problem-solving-algorithmizing-representation	$D_U f(x,y) = (4x + 15x^2y)cos\left(\frac{\pi}{4}\right) + (10y + 5x^3 - 8)sin\left(\frac{\pi}{4}\right)$.	SF23	Procedural
Mp11	Problem-solving-algorithmizing-representation	$sin\ \left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$	SF24	Procedural
Mp12	Problem-solving-algorithmizing-representation	$\boldsymbol{D_U f(x,y)} = (4x + 15x^2y)\left(\frac{\sqrt{2}}{2}\right) + (10y + 5x^3 - 8)\left(\frac{\sqrt{2}}{2}\right)$.	SF25	Procedural
Mp13	Problem-solving-algorithmizing-representation	$D_U f(x,y) = \left[(4x)\left(\frac{\sqrt{2}}{2}\right) + (15x^2y)\left(\frac{\sqrt{2}}{2}\right)\right] + \left[(10y)\left(\frac{\sqrt{2}}{2}\right) + (5x^3)\left(\frac{\sqrt{2}}{2}\right) - 8\left(\frac{\sqrt{2}}{2}\right)\right]$.	SF26	Procedural
Mp14	Problem-solving-algorithmizing-representation-argumentation	$D_U f(x,y) = \left[(2x)(\sqrt{2}) + \frac{(15x^2y\sqrt{2})}{2}\right] + \left[(5y)(\sqrt{2}) + \frac{(5x^3)(\sqrt{2})}{2} - (4\sqrt{2})\right]$.	SF27, SF28, SF29	Procedural
Mp15	Problem-solving-algorithmizing-representation	P59 replaced the point $P = (0, 3)$ in the directional derivative $D_U f(0,3) = \left[(2(0))(\sqrt{2}) + \frac{(15(0)^2(3)\sqrt{2})}{2}\right] + \left[(5(3))(\sqrt{2}) + \frac{(5(0)^3)(\sqrt{2})}{2} - (4\sqrt{2})\right]$.	SF30, SF31	Procedural
Mp16	Problem-solving-algorithmizing-representation-argumentation	$D_U f(0,3) = 15\sqrt{2} - 4\sqrt{2} = 11\sqrt{2}$.	SF32, SF33, SF34	Procedural

example, the students solved the questionnaire with the five proposed tasks, which were qualified from 0 to 5, but for the students who made mistakes in some task(s), their qualification was from 0 to 4.

Figure 17 shows that 72% (*n* = 145) corresponds to students who performed all five tasks correctly. 11% (*n* = 22) refer to the students who obtained 4.0 in the qualification and have an incorrect task. Likewise, 10% (*n* = 20) are the students who achieved the qualification of 3.0, which implies that they have two incorrect assignments. 4% (*n* = 8) are the students who failed in solving three tasks and got the qualification 2.0. The 1% (*n* = 2) reports that two students failed to solve four tasks and their qualification was 1.0. While 2% (*n* = 5) are the students who failed to solve any of the tasks proposed in the questionnaire and their qualification was 0. Furthermore, on the one hand, it is deduced that 7% (*n* = 15) are the students who did not pass the questionnaire because the minimum qualification for passing is 3.0. On the other hand, 93% (*n* = 187) represent the students who passed the questionnaire because they at least solved 3 tasks, or 5 tasks correctly.

Specifically, **Figure 18** shows all the scores (questionnaire qualifications) of the students.

Finally, the report of the results of this study shows a particular case of one of the students who did not solve task 1 properly because they made personal connections and activated an error in the procedure (**Figure 19**).

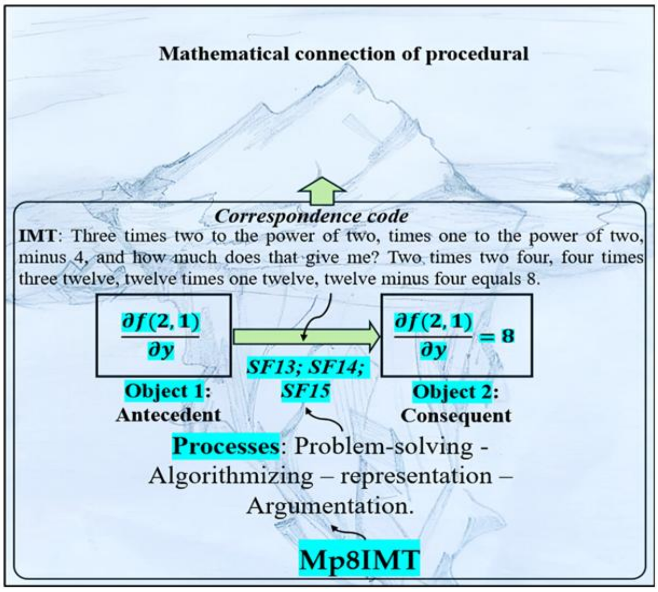

Figure 17. IMT's mathematical connection of procedural (Source: Authors' own elaboration)

DISCUSSION

The results of this research have revealed that the vector calculus teacher explains the topics associated with vectors, partial and directional derivatives based on mathematical connections (instruction-oriented, meaning, procedural, different representations, feature and part-whole), which is evidenced in the step-by-step followed to find the directional derivative.

In addition, in the middle of the explanation, it encourages student participation, a fundamental aspect to ensure the understanding and appropriation of the concepts addressed.

Another fundamental aspect is that most of the students solved the tasks successfully (72% corresponding to 145 students), where they activated mathematical connections of meaning, procedural, different representations, feature, part-whole, which were detailed in terms of mathematical practices, processes/objects and SFs that relate them.

The other 28% of the students (corresponding to 57 students) made at least one error caused by some personal connection made and, therefore, did not solve all five tasks correctly. It is worth noting that only 15 students (7%) failed the questionnaire because they obtained a qualification lower than 3.0, which is the minimum qualification for passing. Although few students failed the questionnaire, as teachers and

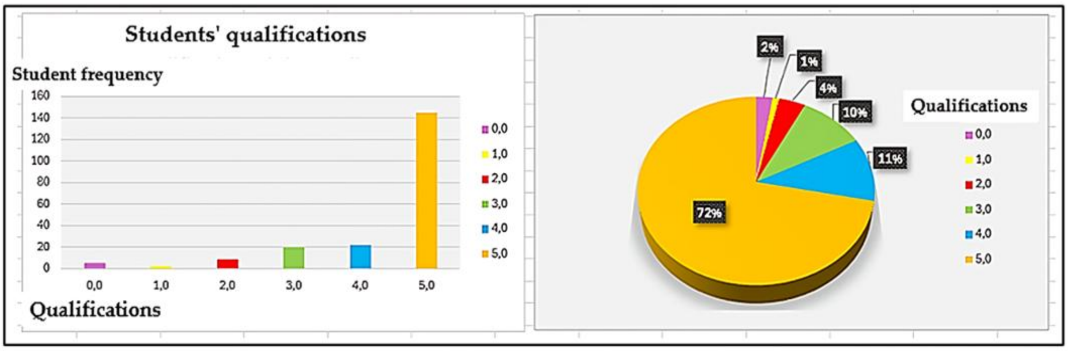

Figure 18. Student qualifications (Source: Authors' own elaboration)

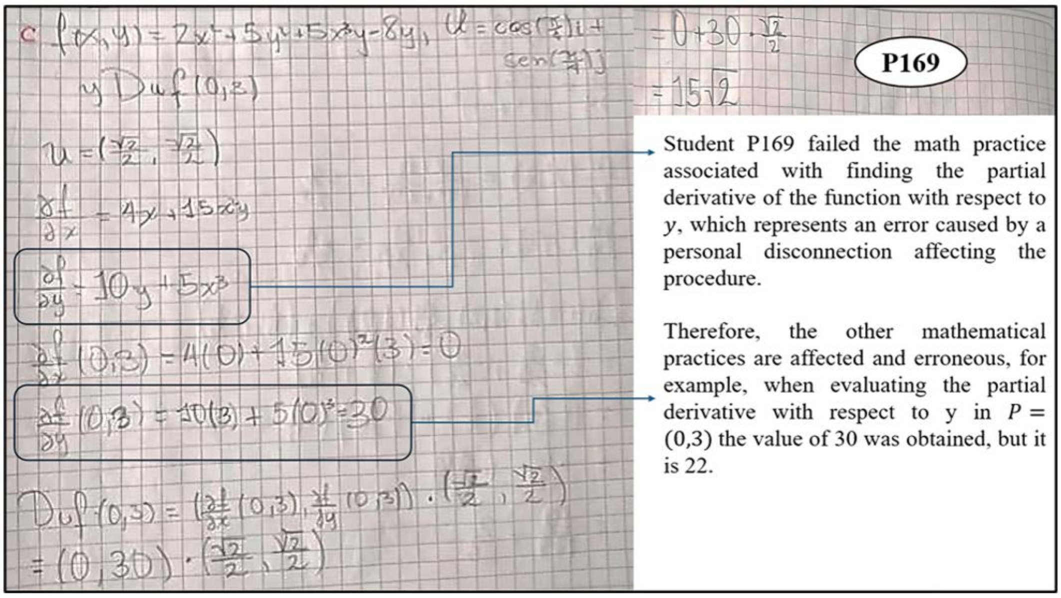

Figure 19. Evidence of personal connection affecting the P169 procedure in task 1 (Source: Authors' own elaboration)

researchers we are committed to continue applying teaching and learning strategies (using mathematical connections) to contribute to the understanding of the concepts of vector calculus and mitigate errors.

One of the most common errors was procedural, such as failing to properly connect steps when calculating partial derivatives (**Figure 19**). Students also struggled with finding curl due to incorrect use of the vector product and determinants involved in partial derivative operations. These challenges highlight the need for teachers to design lessons rich in mathematical processes and representations, including verbal, numerical, graphical, and symbolic languages. Emphasis should be placed on demonstrating definitions, clear procedures, and logically supported propositions, fostering communication and argumentation to support students in overcoming these difficulties.

Students in this research attributed meanings to the concepts of vector, partial derivative, and directional derivative, graphing vectors and the tangent line in a plane on a surface within its vector space (**Figure 6** and **Figure 7**). However, some only associated these concepts with their definition as a limit. Various authors (Bajracharya et al., 2019; MacGee & Moore-Russo, 2015; Martínez-Planell et al., 2015, 2017; Roundy et al., 2015; Thompson et al., 2012; Wangberg & Johnson, 2013; Wangberg et al., 2022; Weber, 2015; Weber et al., 2012) emphasize the importance of geometric representations in teaching partial and directional derivatives, as these enhance mathematical understanding through connections. Moreno-Arotzena et al. (2021) highlight the role of graphical gradient representations in fostering visual understanding. While some students demonstrated strong procedural and geometric connections, others struggled with partial derivatives. As suggested by Weber (2015) and Weber et al. (2012), mastering partial derivatives is crucial for correctly solving directional derivatives.

We are convinced that these difficulties of students in solving problems with partial derivatives, directional derivatives, gradient, curl, among other concepts, are caused because students still have difficulties in understanding the concept of vector, doing operations with vectors, finding the norm, the vector and scalar product and finding derivatives graphically, etc. (Barniol & Zavala, 2016; Flores-García et al., 2007; Possani et al., 2010; Rakkapao et al., 2016; Rodríguez-Nieto et al., 2024; Rodríguez-Vásquez et al., 2024; Salgado & Trigueros, 2014; Susac et al., 2018).

Regarding the theoretical and methodological aspects, it was evident that the data were analyzed in an organized and detailed manner, assessing the mathematical practices, the configurations of primary objects and the connections identified thanks to the networking between the ETC and the OSA. This is an analysis method that can be used in different scenarios and works where analysis of mathematical activity and understanding of the concepts is required. One of the limitations of this research is that the results cannot be generalized due to their qualitative and descriptive nature, and it is advisable to perform a quantitative analysis. In addition, this study did not use software to graph or create procedures, which is important for calculus classes and active mathematical connections.

CONCLUSIONS

This research revealed a teaching methodology by the IMT on vector calculus, focused on mathematical connections, which has been effective in the understanding and resolution of mathematical problems on partial and directional derivation (including rotation and divergence) by the participating students. Also, in this work, the active participation of the students in the classes was promoted and was fundamental to ensure the assimilation of the concepts addressed.

The analysis shows details about the mathematical connections that activate a type of understanding of the concepts of vector calculus, but also suggests that difficulties in understanding concepts such as vector, partial and directional derivative, gradient and curl are persistent among some students (at least in this research there are 15 students who urgently need attention because they did not reach the minimum qualifications of 3.0 to pass the questionnaire), which highlights the need to improve the instruction and use of geometric representations. In fact, previous studies mark the frontier of research with this problem and agree on the importance of working with graphic representations to facilitate the visual and conceptual understanding of these topics. In addition, procedural understanding should be strengthened with students because to find the gradient, the directional derivative and curl, the partial derivative is required, which is the concept that generates problems in students, not only for derivatives or functions, but for arithmetic and algebraic aspects implicit in the norm, unit vector, dot and vector product, among other operations with vectors.

Future studies should use exhaustive and detailed data analysis methods focused on mathematical practices and onto-semiotic configurations. This approach, emphasizing mathematical connections, can be applied in various educational contexts. The study also highlights the potential for developing didactic strategies that strengthen understanding through connections, geometric representations, and argumentation, linking institutional mathematics to daily life, reducing errors, and improving academic performance.

Author contributions: CAR-N: conceptualization, methodology, supervision, and writing–original draft & **VFM:** formal analysis, validation, writing–original draft, writing–review & editing, and resources. Both authors agreed with the results and conclusions.

Funding: This study is part of the projects, Proyecto de docencia codificado por DOC.100-11-001-18 (Universidad de la Costa) & Grant PID2021-127104NB-I00 funded by MICIU/AEI/10.13039/501100011033 and by "ERDF A way of making Europe".

Acknowledgments: This study is part of the projects: Proyecto de docencia codificado por DOC.100-11-001-18 and Ministerio de Ciencia e Innovación de España PID2021-127104NB-I00.

Ethical statement: The authors stated that the participants and the university were informed that this article is for educational purposes and not for economic or political purposes and they agreed to participate voluntarily. In addition, one of the researchers is a full-time professor with research commitments at the university as mentioned in the participants and context section and has authorization to conduct research with the student population with the purpose of improving students' understanding of mathematical concepts.

Declaration of interest: No conflict of interest is declared by the authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

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