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Characterization of sub-fields of derivative problems in engineering textbooks

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Abstract

This paper aims to characterize the subfields of problems emerging from problem-situations of the derivative in university textbooks proposed for the training of civil and commercial engineers. Thirteen textbooks used for the teaching of the derivative in the syllabi of different Chilean universities were qualitatively analyzed. The notion of epistemic configuration of the onto-semiotic approach to mathematical knowledge and education was used for this purpose. It is observed that a complex approach to the derivative is adopted in the textbooks considered. A total of 21 subfields of problems appear, although not all the textbooks extensively use subfields of derivative problems. The findings of this research establish relevant guidelines for the design of a specific didactic proposal to learn the construction of the derivative in engineering courses.

Keywords: teaching the derivative, onto-semiotic approach, engineering, textbooks

INTRODUCTION

Mathematics, understood as a basic science, is fundamental for all areas of knowledge. The derivative mathematical object is key for engineers, especially civil and commercial engineers. Not only does it allow the study of the behavior of functions and variational analysis, but also the optimization of resources and marginal analysis, amongst other aspects. Moreover, its different ways of representation make it one of the most widely used mathematical objects in microeconomics, especially in topics related to understanding economic and mathematical concepts (Ballard & Johnson, 2004; Butler et al., 1998; García et al., 2006; Hey, 2005). They reveal difficulties in the interpretation of economic situations due to the poor understanding of the mathematical meanings that organize them (Ariza & Llinares, 2009).

As a consequence of its various applications, the derivative has become one of the fundamental mathematical objects present in the educational processes of the different engineering disciplines. Several studies have been conducted on the complexity of its meanings, its multiple representations, the teaching and learning processes of engineering students, the appropriateness of the meaning of the derivative in the different curricula, and the partial meanings in university teaching textbooks for engineering (Galindo Illanes & Breda, 2023, 2024; Galindo Illanes et al., 2022; Larios & Jiménez, 2022; Larios et al., 2021; Pino-Fan et al., 2018; Rodríguez-Nieto et al., 2022, 2023). The organization of the components in which this complexity is clear is present in almost all emerging theoretical frameworks in the area of mathematics education. Working with the different meanings of a mathematical object is an aspect proposed by the onto-semiotic approach (OSA) to mathematical knowledge and education (Godino et al., 2007), which proposes to analyze the complexity of mathematical objects through their pluri-meanings (partial meanings). One of the central elements of the complexity of the derivative is framed in the partial meanings of the problem-situations proposed for the teaching and learning of this mathematical object.

One of the resources that most influences the construction of the meaning of a mathematical object are the textbooks considered in the curricula of institutions. Teachers use them to organize and implement lessons, and they are hence part of the students' educational process (Larios & Jiménez, 2022). The analysis of textbooks does not replace classroom observation.

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Contribution to the literature

- Use of the epistemic configuration of the OSA in a detailed way for the configuration of new problem fields (PFs) on the derivative.
- Emergence of 21 problem subfields of the derivative applied to engineering courses.
- The wide variety of problem subfields allows the design of didactic proposals for the learning of the derivative in engineering courses.

However, it can provide information to build assessment instruments and to improve teaching (Alvarado & Batanero, 2008). Pino-Fan et al. (2013) found five PFs of the derivative in an analysis of problem situations of the mathematical object present in textbooks in Mexico. They are summarized as PFs on tangents; on calculating instantaneous rates of change; on instantaneous variance; on applying the derivative for calculating maxima and minima, the analysis of graphs of functions etc.; and on calculating derivatives from derivative rules and theorems. Galindo Illanes and Breda (2023) adopted these five PFs to analyze the complexity of the problem-situations that appear in the textbooks for commercial engineering courses in Chile. However, they did not perform an in-depth analysis of this complexity based on the refinement of the problem-situations of the derivative present in these teaching textbooks.

With the aim of conducting a study of the categorization of the problem-situations of the derivative in textbooks, based on the extension of the research carried out in Pino-Fan et al. (2013) and Galindo Illanes and Breda (2023), it is of great interest to investigate how the meanings of the derivative are organized in the problem-situations in the textbooks used in engineering courses. In this regard, this paper seeks to characterize the problem subfields of the problem-situations of the derivative in the university textbooks proposed for the training of civil and commercial engineers.

THEORETICAL FRAMEWORK

The theoretical perspective adopted in this research is the OSA (Godino et al., 2007, 2019). Font et al. (2013) explain that the notion of complexity of the mathematical object and the organization of the components of this complexity play a key role. Understanding complexity, in terms of a plurality of meanings, is a result of the pragmatist view of meaning assumed in the OSA. From a pragmatist point of view, the meaning of a mathematical object is understood as the set of practices in which that object intervenes in a decisive manner (or not). A mathematical object, which has originated emerging from the system of practices that allows a certain PF to be solved, is framed over time in different research programs. Each new research program enables solving new types of problems, applying new procedures, linking the object (and, therefore, and defining it) in a different manner, using new representations, etc. Hence, in the course of time, new subsets of practices appear that expand the meaning of the object.

In the OSA, a mathematical activity is assumed to be a human activity centered on problem-solving, taking place in a given time-space, through a sequence of practices that are often considered processes (of meaning, conjecturing, arguing, etc.). The OSA therefore proposes the notion of problem-situation of mathematical practice (sequence of practices) that takes place during the resolution of these problem situations. In particular, the use and/or emergence of the primary objects of the configuration (problem situations, language, procedures, concepts/definitions, propositions and arguments) occur throughout the respective mathematical processes of communication, problematization, definition, enunciation, development of procedures (creation of algorithms and routines), and argumentation (applying the process-product duality).

For the derivative mathematical object, Pino-Fan et al. (2011) characterized its complexity through nine configurations of primary objects:

(1) tangent in Greek mathematics,

(2) variation in the Middle Ages,

(3) algebraic methods to find tangents,

(4) cinematic conceptions for the tracing of tangents,

(5) intuitive ideas of limits for calculating maxima and minima,

(6) infinitesimal methods in the calculation of tangents,

(7) calculation of fluxions,

(8) calculation of differences, and

(9) the derivative as limit.

In Pino-Fan et al. (2013) these nine configurations are used to reconstruct the global meaning of the derivative, which is employed to assess the representativity of the intended meaning in the Mexican baccalaureate curriculum (based on the configurations of the primary objects activated in the mathematical practices proposed both in the curriculum and in textbooks of that level). With respect to the PFs of the derivative present in the textbooks, Pino-Fan et al. (2013) propose five categories.

Of the types of primary mathematical objects present in the OSA, this paper considers problem-situations for the analysis of the derivative. Problem-situations are understood as phenomenological situations that give

Table 1. Engineering-oriented mathematics textbooks (Ts)

No	Title	Authors	Publisher	Edition
T1	Matemáticas aplicadas a la administración y la economía [Mathematics applied to administration and economics]	Arya and Lardner (2009)	Pearson	5th
T2	Matemáticas para administración y la economía [Mathematics for management and economics]	Haeussler et al. (2008)	Pearson	12th
T3	Cálculo para administración, economía, ciencias biológicas y sociales [Calculus for administration, economics, biological and social sciences]	Hoffmann et al. (2006)	McGraw Hill	8th
T4	Matemáticas aplicadas para administración, economía y ciencias sociales [Applied mathematics for administration, economics and social sciences]	Budnick (2007)	McGraw Hill	4th
T5	Matemáticas para administración y economía [Mathematics for administration and economics]	Tan (2005)	Thomson	3rd
T6	Teoría y problemas de cálculo para administración, economía y ciencias sociales [Theory and problems of calculus for administration, economics and social sciences]	Dowling (1992)	McGraw Hill	1st
T7	Thomas' cálculo una variable [Thomas' one variable calculus]	Thomas et al. (2015)	Pearson	13th
T8	Cálculo diferencial e integral [Differential and integral calculus]	Purcell et al. (2007)	Pearson	9th
T9	Cálculo tomo 1 [Calculus volume 1]	Larson and Edwards (2016)	McGraw Hill	10th
T10	Cálculo trascendentes tempranas [Early transcendental calculus]	Stewart (2018)	Cengage Learning	8th
T11	El cálculo [The calculus]	Leithold (1998)	OPU	7th
T12	Cálculo diferencial [Differential calculus]	Ortiz (2014)	GEP	1st
T13	Cálculo diferencial e integral [Differential and integral calculus]	Aguilar et al. (2010)	Pearson	1st

Note. OUP: Oxford University Press & GEP: Grupo Editorial Patria

rise to mathematical activities and can be categorized into types or PFs. To this end, the presence of subfields of problems of the major PFs of the derivative in a sample of textbooks based on the five categories proposed in Pino-Fan et al. (2013) are analyzed.

METHODOLOGY

The study context, the data collection instruments, and the data analysis are explained in this section.

Context of the Study and Data Collection Instruments

Textbooks are considered a very important resource in the teaching and learning of mathematics. Teachers use the ones that appear in the bibliographies of the programs of study to plan and implement their lessons (Galindo Illanes & Breda, 2023). This means textbooks influence the teacher's teaching and the students' learning. They hence occupy a crucial role in the determination of the intended meaning.

Following Galindo Illanes and Breda (2023), thirteen engineering-oriented mathematics textbooks that provide a broad spectrum of the meaning of the derivative, were considered. They are classified into six mathematics textbooks applied to commercial engineering, and seven classic textbooks on the teaching of calculus for civil engineering (see **Table 1**).

The steps followed for the content analysis of textbooks are those established by Alvarado and Batanero (2008), which can be summarized, as follows:

- ✓ determine the university mathematics textbooks to be analyzed,
- ✓ select the chapters by means of a detailed reading of the ones that address the subject,
- ✓ determine the elements of meaning, specifically the PFs, based on the epistemic analysis,
- ✓ draw up comparative tables showing the PFs in the different textbooks selected,
- ✓ carry out comparative content analysis, and
- ✓ present the conclusions by means of a descriptive analysis of the data obtained.

Based upon the 13 textbooks, the organization of the contents was identified, and the chapters addressing the derivative were selected. Next, the presence of the PFs of the historical study conducted by Pino-Fan et al. (2013) was analyzed. The PFs are summarized, as follows:

(1) PFs on tangents,

(2) PFs on calculating instantaneous rates of change (referring to the quotient between two quantities of magnitude),

(3) PFs on instantaneous variance (referring to the quotient between two numbers without reference to quantities of magnitude, commonly known as the incremental quotient limit),

(4) PFs on applying the derivative for calculating maxima and minima, the analysis of graphs of functions, etc., and

(5) PFs on calculating derivatives from derivative rules and theorems.

Table 2. PFs, identifier (ID), and description of the PF (Galindo Illanes & Breda, 2023)

PF	ID	Description of the PF
A	A1	Calculation of the slope of a tangent line to a curve at a point using the Cartesian notion
	A2	Calculation of the slope of a tangent line tangent to a curve at a point using Leibniz' triangle
	A3	Calculation of the equation of a tangent line to a curve at a point using the derivative
	A4	Calculation of the slope of a tangent line to a curve to obtain an instantaneous rate of change
B	B1	Calculation of instantaneous rates of marginal change
	B2	Population growth
	B3	Calculation of instantaneous speed
	B4	Relative and percentage rates of change for economic functions
	B5	Rates of change applied to other areas of knowledge
C	C1	Instantaneous variance of real functions
	C2	Relative and percentage rates of change for real functions
D	D1	Analysis of the graph of real functions (monotonicity) and of areas of knowledge other than economics
	D2	Analysis of the graph of economic functions (monotonicity)
	D3	Calculation of local maxima and minima in applied economics problems using the criterion of the first or second derivative for relative extrema
	D4	Calculation of local maxima and minima in real functions and in areas of knowledge other than economics, using the criterion of the first or second derivative for relative extrema
	D5	Absolute extrema on a closed interval
	D6	Analysis of concavity of real and economic functions
E	E1	Calculation of higher order derivatives
	E2	Calculation of implicit derivatives
	E3	Newton's method for approximating roots of polynomials
	E4	L'Hôpital's rule for calculating limits

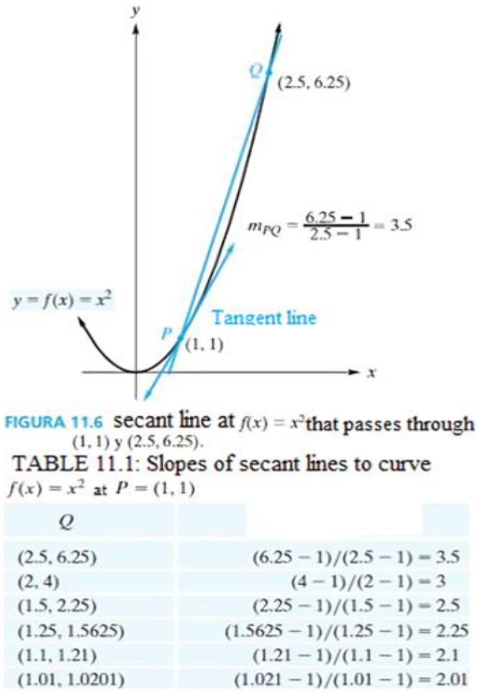
Through the analysis of the PFs in the textbooks, 21 subfields of Galindo Illanes and Breda (2023) problems were identified (**Table 2**). This first analysis enables showing and giving examples of the emergence of each PF subcategory and carrying out a detailed characterization of the PFs of the derivative in the textbooks.

RESULTS

Manuscript should be typed using word processors (Microsoft Word or Open Office) software. The results of the analysis, which provide specific examples that clarify the different subfields of problems encountered in the PFs on the derivative in textbooks, are presented below.

PFs on Tangents

To analyze the presence of this PF, attention needs to be paid to the construction of the tangent line to a curve considered in the textbooks on the concept. It is interesting to identify if the type of activities, exercises, and/or examples in the textbooks allow extending the Euclidean notion of the tangent line to the Cartesian and Leibnizian ones, considering the infinitesimal methods, since the calculation leads students to the highest levels of generality of the tangent (Santi, 2011). Examples of four subfields of this PF present in the textbooks are shown below. The objective is to understand the geometric interpretation of the derivative, previously considering the concept of tangent line (Galindo Illanes & Breda, 2023, 2024).



Q	
(2.5, 6.25)	(6.25 − 1)/(2.5 − 1) = 3.5
(2, 4)	(4 − 1)/(2 − 1) = 3
(1.5, 2.25)	(2.25 − 1)/(1.5 − 1) = 2.5
(1.25, 1.5625)	(1.5625 − 1)/(1.25 − 1) = 2.25
(1.1, 1.21)	(1.21 − 1)/(1.1 − 1) = 2.1
(1.01, 1.0201)	(1.021 − 1)/(1.01 − 1) = 2.01

Figure 1. Example of A1 present in T2 (Haeussler et al., 2008, p. 482)

1. *Calculation of the slope of a tangent line to a curve at a point using the Cartesian notion (approximation by secant lines)*

This type of task is commonly used to construct the algebraic definition of the derivative at a point through its geometric interpretation. To do so, the Cartesian notion of the slope of a tangent line to a curve at a point is considered the limit of slopes of secant lines. Its

Example 5. The slope of a curve $f(x) = x^2$ is found by substituting the specific function into the algebraic formula (4.1): (a) substituting the specific function in the algebraic formula, (b) simplifying the function, and (c) evaluating the limit of the function in its simplified form.

From (4.1): Slope $T = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$.

(a) Substituting $f(x) = x^2$, Slope $T = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$.

(b) Simplifying the result, Slope $T = \lim_{\Delta x \to 0} \frac{x^2+2x(\Delta x)+(\Delta x)^2-x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x(\Delta x)+(\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(2x+\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} 2x + \Delta x$.

(c) Find the limit of the simplified expression, Slope $T = 2x$.

Note that the value of the slope depends on the value chosen for x.

Figure 2. Example of A1 present in T6 (Dowling, 1992, p. 99)

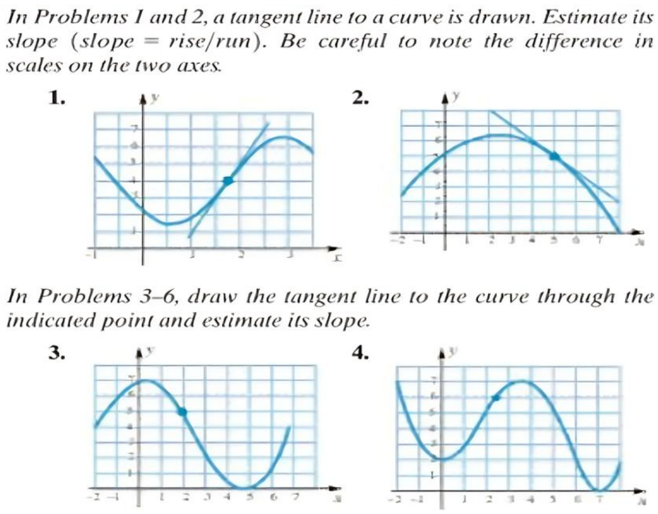


Figure 3. Example of A2 present in T8 (Purcell et al., 2007, p. 97)

A finding equation of a tangent line in exercise 25-32,
(a) find an equation of the tangent line to the graph of f at the given point,
(b) use a graphing utility to graph the function and its tangent line at the point, and
(c) use the derivative feature of a graphing utility to confirm your results.

25. $f(x) = x^2 + 3, (-1, 4)$	29. $f(x) = \sqrt{x}, (1, 1)$
26. $f(x) = x^2 + 2x - 1,$ (1, 2)	30. $f(x) = \sqrt{x-1}$ (5, 2)
27. $f(x) = x^3, (2, 8)$	31. $f(x) = x + \frac{4}{x}, (-4, -5)$
28. $f(x) = x^3 + 1, (-1, 0)$	32. $f(x) = \frac{6}{x+2}, (0, 3)$

Figure 4. Example of A3 present in T9 (Larson & Edwards, 2016, p. 103)

development is commonly tabular (calculation of slopes of nearby secant lines) with graphical support. An example is shown in **Figure 1**.

Another typical development of this PF present in the textbooks is to determine the slope of the tangent line to the curve at a point using its algebraic definition (limit calculation), to then move on to the definition of the derivative of a function at a point. A standard example is shown below. Its solution, which is proposed in the textbook, uses the algebraic definition of the slope of the tangent line $m_{tan} = \lim_{h \to 0} \frac{f(x_1+h)-f(x_1)}{h}$. **Figure 2** provides an example.

2. *Calculation of the slope of a tangent line to a curve at a point using Leibniz's triangle*

This type of task provides the graph of the curve, which is usually represented in a grid. A ruler is used to determine an approximation of the slope of the tangent line at a point.

Moreover, it allows identifying a tangent line to a curve at a point in a graph. Leibniz's differential triangle is used to determine an approximation of the slope. An example is shown in **Figure 3**.

3. *Calculation of the equation of the tangent line to a curve at a point using the derivative*

Find the equations of the tangent lines: Sketch the graphs of $y = x^2$ and $y = -x^2 + 6x - 5$, and sketch the two lines that are tangent to both graphs. Find equations of these lines.

Figure 5. Example of A3 present in T9 (Larson & Edwards, 2016, p. 115)

This is a type of task that allows strengthening the geometric interpretation of the derivative function, as well as to understand, from the notion of process, that the slope of a tangent line to a curve at a point corresponds to the derivative of the curve evaluated at the point of tangent. In the examples provided below, in order to find the equation of the tangent line, the derivative at the point of tangency must first be found. In both examples, the derivative is first calculated. In the first one (**Figure 4**), the algebraic definition of the derivative is used, and, in the second (**Figure 5**), the derivative rules are used.

Subsequently, in both examples, the derivative is evaluated at the point of tangency. The examples are presented in **Figure 4** and **Figure 5**.

4. *Calculation of the slope of a tangent line to a curve to obtain the instantaneous rate of change*

This type of problem-situation allows applying the tangent line to a curve as the instantaneous rate of

The table shows values of the viral load $V(t)$ in HIV patients 303, measured in RNA copies/mL, t days after ABT-538 treatment was begun.

t	4	8	11	15	22
$V(t)$	53	18	9.4	5.2	3.6

(a) Find the average rate of change of V with respect to t over each time interval:

i. $[4, 11]$ iii. $[11, 15]$

ii. $[8, 11]$ iv. $[11, 22]$

(b) Estimate and interpret the value of the derivative $V'(11)$.

Figure 6. Example of A4 present in T10 (Stewart, 2018, p. 150)

change. We then move on to the geometric interpretation of the derivative. It is common to see that this type of exercise considers physics problems in textbooks. **Figure 6** shows an example.

PFs on Calculating Instantaneous Rates of Change

In order to understand and model certain economic situations, it is essential for students to be able to extract the information provided by the graphical representations of the relationships between variables and the measurement of the variance of change (Ariza & Llinares, 2009). It is hence important to consider the idea of variance and change when constructing the derivative concept (Zambrano et al., 2019), and to incorporate variational elements and give meaning to the different elements related to variance (Vrancken & Engler, 2014).

To establish instantaneous rates of change, understood as the "quotient" between two quantities of magnitude, an average rate of change is initially used, for example, $v_{average} = \frac{displacement}{total\ time\ taken} = \frac{\Delta s}{\Delta t}$ (average rate of change of s with respect to t), which is developed tabularly in order to introduce the incremental notation and subsequently the differential notation $\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$, . Finally, the derivative is represented as $s'(t) = \frac{ds}{dt}$ (Inglada & Font, 2003). In the following, examples of the five subfields of this PF are considered, which enables understanding and modelling certain applied situations.

1. *Calculation of instantaneous rates of marginal change (marginal cost, marginal revenue, marginal utility, marginal productivity, marginal return, marginal tax rate, and marginal propensity to save and consume)*

To begin the analysis of the presence of this subcategory, it is worth mentioning that the textbooks algebraically define marginal concepts in economics using the concept of rate of change. For example,

> "A manufacturer found that the total cost function c=f(q) gives the total cost c of producing and marketing q units of a product. The rate of change of c with respect to q is called marginal cost. Thus, $Marginal\ cost = \frac{dc}{dq}$" (Haeussler et al., 2008, p. 501).

It should be mentioned that the proposals of the textbooks consider the calculation of the derivative

Economics

23. Marginal cost Suppose that the dollar cost of producing x washing machines is $c(x) = 2000 + 100x - 0.1x^2$.

a. Find the average cost per machine of producing the first 100 washing machines.

b. Find the marginal cost when 100 washing machines are produced.

c. Show that the marginal cost when 100 washing machines are produced is approximately the cost of producing one more washing machine after the first 100 have been made, by calculating the latter cost directly.

24. Marginal revenue Suppose that the revenue from selling x washing machines is

$$r(x) = 20{,}000\left(1 - \frac{1}{x}\right)$$

dollars.

a. Find the marginal revenue when 100 machines are produced.

b. Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing production from 100 machines a week to 101 machines a week.

Figure 7. Example of B1 present in T7 (Thomas et al., 2015, p. 136)

Exercise 86. In section 1.4 we modeled the world population from 1900 to 2010 with the exponential function $P(t) = (1{,}436.53) \times (1.01395)^t$, where $t = 0$ corresponds to the year 1900 and $P(t)$ is measured in millions. According to this model, what was the rate of increase of world population in 1920? In 1950? In 2000?

Figure 8. Example of B2 present in T10 (Stewart, 2018, p. 206)

function in their developments. Although the method of obtaining it is not specified, it is observed that most of them use derivative rules. **Figure 7** provides an example.

2. *Population growth*

To find the rate of change of population growth over time, the textbooks use the derivative of the function representing a number of people over time, evaluated at the time in question. This kind of exercise is usually proposed in sections at the end of a chapter, which can be either at the rate of change, or on derivative rules. An example is given in **Figure 8**.

3. *Calculation of instantaneous speed*

To find the instantaneous speed of a moving object, the derivative of the position function with respect to time, evaluated at the time in question, is used. Usually, this type of exercise is proposed in two places in the

Example 4. An object, initially at rest, falls due to gravity. Find its instantaneous velocity at t = 3.8 seconds and at t = 5.4 seconds.
Solution. Calculate the instantaneous velocity at $t = c$ seconds. As f(t) = 16t²,

$$v = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$
$$= \lim_{h\to 0}\frac{16(c+h)^2-16c^2}{h}$$
$$= \lim_{h\to 0}\frac{16c^2+32ch+16h^2-16c^2}{h}$$
$$= \lim_{h\to 0}(32c+16h) = 32c$$

The instantaneous velocity at 3.8 seconds is 32(3.8) = 121.6 feet per second. At 5.4 seconds it is 32(5.4) = 172.8 feet per second.

Figure 9. Example of B3 present in T8 (Purcell et al., 2007, p. 94)

Example 2.2.7. The gross domestic product (GDP) of a certain country was $N(t) = t^2 + 5t$ + \$106 billion t years after 1995.
(a) At what rate was the GDP changing with respect to time in 2003?
(b) At what percentage rate was the GDP changing with respect to time in 2003?
Solution.
(a) The rate of change of GDP is the derivative $N'(t) = 2t + 5$. The rate of change in 2003 was $N'(8) = 2(8) + 5$ = \$21 billion per year.
(b) The percentage rate of change of GDP in 2003 was $100\frac{N'(8)}{N(8)} = 100\frac{21}{210}$ = 10% per year.

Figure 10. Example of B4 present in T3 (Hoffmann et al., 2006, p. 114)

Physical chemistry: According to Debye's formula in physical chemistry, the orientation polarization P of a gas satisfies $P = \frac{4}{3}\pi N\left(\frac{\mu^2}{3kT}\right)$, where μ, k, and N are positive constants, and T is the temperature of the gas. Find the rate of change of P with respect to T.

Figure 11. Example of B5 present in T3 (Hoffmann et al., 2006, p. 120)

textbooks, either in the introduction of a chapter to construct the derivative at a point as the instantaneous rate of change, or at the end of a chapter, which can either be on the rate of change or on derivative rules. **Figure 9** provides an example.

4. *Relative and percentage rates of change for economic functions (percentage rate of change of income as a function of percentage change in units or price, percentage rate of change of quantity as a function of percentage change in price [elasticity of demand])*

The textbooks algebraically define the concepts of relative and percentage rates for economic functions using the concept of rate of change. For example,

> "Definition: The relative rate of change of $f(x)$ is $\frac{f'(x)}{f(x)}$. The percentage rate of change of $f(x)$ is $\frac{f'(x)}{f(x)} * 100$" (Haeussler et al., 2008, p. 504).

These kinds of problems are usually proposed in sections at the end of the chapter on the derivative as a rate of change as homework for the reader. An example is presented in **Figure 10**.

Example 3. Finding a rate of change. Find the rate of change of $y = x^4$ with respect to x and evaluate it when x = 2 and when x = -1. Interpret the results.
Solution. The rate of change is $\frac{dy}{dx} = 4x^3$ when $x = 2$, $dy/dx = 4(2)^3 = 32$. This means that if x increases, from 2 onwards, by a small amount, y grows about 32 times that amount. Or put more simply, it is said that when $x = 2$, y grows 32 times faster than x. When $x = -1$, $dy/dx = 4(-1)^3 = -4$. The meaning of the minus sign in -4 is that y decreases at a rate 4 times faster than x increases.

Figure 12. Example of C1 present in T2 (Haeussler et al., 2008, p. 500)

Example 9. Relative and percentage rates of change. Find the relative and percentage change rates of $y = f(x) = 3x^2 - 5x + 25$ when $x = 5$.
Solution. Here $f'(x) = x - 5$. As $f'(5) = 6(5) - 5 = 25$ and $f(5) - 3(5)^2 - 5(5) + 25 = 75$, the relative rate of change of y when $x = 5$ is $\frac{f'(5)}{f(5)} = \frac{25}{75} = 0.333$. Multiplying 0.333 by 100% gives the percentage rate of change: (0.333)(100%) = 33.3%.

Figure 13. Example of C2 present in T2 (Haeussler et al., 2008, p. 504)

Example 1. Find the interval on which function $f(x) = x^2$ is increasing and the interval on which it is decreasing.

Figure 14. Example of D1 present in T5 (Tan, 2005, p. 628)

The demand for the product of a firm varies with the price that the firm charges for the product. The firm estimates that annual total revenue R (stated in \$1,000s) is a function of the price p (stated in dollars). Specifically, **[05]**

$$R = f(p) = -50p^2 + 500p$$

(a) Determine the price which should be charged in order to maximize total revenue.
(b) What is the maximum value of annual total revenue?

Figure 15. Example of D3 present in T5 (Budnick, 2007, p. 813)

5. *Rates of change applied to other areas of knowledge*

These problems are typically proposed in sections at the end of the chapter on the derivative as a rate of change as homework for the reader. **Figure 11** offers an example.

PFs on Instantaneous Variance

This PF refers to the quotient between two numbers without reference to quantities of magnitude. It is commonly known as the incremental quotient limit. **Figure 12** and **Figure 13** offer examples of the two subfields of this PF, which allow the concept of variance to be applied in different contexts.

1. *Instantaneous variance of real functions*
2. *Relative and percentage rates of change for real functions*

PFs on Applying the Derivative for Calculating Maxima and Minima, the Analysis of Graphs of Functions, etc.

Figure 14 and **Figure 15** are examples of two subfields of this PF. It enables applying the concept of

Finding a second derivative. In exercises 91 to 98, find the second derivative of the function.

91. $f(x) = x^4 + 2x^3 - 3x^2 - x$ **92.** $f(x) = 4x^5 - 2x^3 + 5x^2$

93. $f(x) = 4x^{3/2}$ **94.** $f(x) = x^2 + 3x^{-3}$

95. $f(x) = \frac{x}{x-1}$ **96.** $f(x) = \frac{x^2+3x}{x-4}$

97. $f(x) = x \operatorname{sen} x$ **98.** $f(x) = \sec x$

Figure 16. Example of E1 present in T9 (Larson & Edwards, 2016, p. 127)

Find the derivative y' of the implicit function $\sin(x+y) = x$.
Solution.

$$\frac{d}{dx}\operatorname{sen}(x+y) = \frac{dx}{dx} \quad \rightarrow \quad \cos(x+y)(1+y') = 1 \quad \rightarrow \quad \cos(x+y) + y'\cos(x+y) = 1$$

$$y'\cos(x+y) = 1 - \cos(x+y),$$

where the derivative

$$y' = \frac{1-\cos(x+y)}{\cos(x+y)} = \frac{1}{\cos(x+y)} - \frac{\cos(x+y)}{\cos(x+y)} = \sec(x+y) - 1$$

Figure 17. Example of E2 present in T9 (Aguilar et al., 2010, p. 128)

derivative to the analysis of the monotonicity of functions in different contexts and obtaining relative extrema.

1. *Analysis of the graph of real functions (monotonicity) and of areas of knowledge other than economics*
2. *Calculation of local maxima and minima in applied economics problems using the criterion of the first or second derivative for relative extrema*

PFs on Calculating Derivatives from Derivative Rules and Theorems

Examples of the subfields of this PF, such as the application of the derivative concept through its derivation rules to the calculation of higher order derivatives (**Figure 16**), the calculation of implicit derivatives (**Figure 17**), the approximation of roots (**Figure 18**), and limit theory (**Figure 19**) are presented below.

1. *Calculation of higher order derivatives*
2. *Calculation of implicit derivatives*
3. *Newton's method for approximating roots of polynomials. Construction by means of tangent lines*
4. *L'Hôpital's rule for calculating limits*

The subfields of problems present in the 13 textbooks analyzed are shown in **Table 3**.

With regard to the PFs, all mathematics textbooks applied to economics and business include PFs on the derivative that consider the calculation of the equation of the tangent line to a curve at a point. The presence of the calculation of instantaneous rates of change, marginal, population, relative and percentage rates, especially considering real and applied functions in this area of knowledge is also observed. The analysis of monotonicity, concavity, and relative extrema of real functions and those applied to economics, using the criteria of first and second derivatives also appear.

Example 2. Approximating a root using Newton's method. *Approximate the root of $x^3 = 3x - 1$, which is found between -1 and -2. Continuing the process until two successive approximations differ by less than* 0.0001.
Solution. It is established that $f(x) = x^3 - 3x + 1$ (it is necessary to have the form $f(x)$) is found at $f(-1) = (-1)^3 - 3(-1) + 1 = 3$ and $f(-2) = (-2)^3 - 3(-2) + 1 = -1$ (note the change in sign). Since $f(-2)$ is closer to zero than $f(-1)$, -2 is chosen as the first approximation, $x1$,. Now, $f'(x) = 3x^2 - 3$ so that

$$f(x_n) = x_n^3 - 3x_n + 1 \quad \text{y} \quad f'(x_n) = 3x_n^2 - 3$$

By substituting in equation (4), the recursive formula is obtained

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 3x_n + 1}{3x_n^2 - 3} \quad \text{so that}$$

$$x_{n+1} = \frac{2x_n^3 - 1}{3x_n^2 - 3}$$

Since $x_1 = 2$, determining that $n = 1$ in equation (6) yields

$$x_2 = \frac{2x_1^3 - 1}{3x_1^2 - 3} = \frac{2(-2)^3 - 1}{3(-2)^2 - 3} \approx -1.88889$$

If continued this way, table 12.2 is obtained. Since the values of x_3 and x_4 differ by 0.00006, which is less than 0.0001, then the root is taken to be -1.87939 (this is x_4).

Figure 18. Example of E3 present in T2 (Haeussler et al., 2008, p. 556)

Example 1. The following limits involve indeterminate 0/0 forms, so we apply l'Hôpital's rule. In some cases, it must be applied more than once.
(a) $\lim_{x\to 0} \frac{3x - senx}{x}$, (b) $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$, and (c) $\lim_{x\to 0} \frac{\sqrt{1+x}-1-x/2}{x^2}$.

Figure 19. Example of E4 present in T2 (Haeussler et al., 2008, p. 404)

All classic calculus textbooks for engineering include PFs that consider the calculation of the equation of the tangent line to a curve at a point using derivatives, the calculation of instantaneous rates of change of functions applied to other areas of knowledge, and the analysis of monotonicity, and relative extrema of real functions and functions applied to other areas of knowledge, other than economics. In addition, they consider the calculation of higher order derivatives and implicit functions using derivative rules and theorems.

Finally, it is observed that the main differences between the mathematics textbooks applied to economics and business and the classic textbooks for engineering are found in the subfields of problems of PFs B4, C2, D2, E2, E3, and E4. Moreover, in the classic textbooks, the number of problems and tasks applied to the area of economics and business is greatly reduced, and the emphasis on the partial meaning of the "derivative as a limit" (incremental quotient limit) is noticeable.

Most textbooks consider introductory problems applied to physics, economics, chemistry, pandemics, etc. They also contain solved problems, in which the use of the concepts is exemplified and the proof of some theorems of the derivative is shown. The language used is mostly algebraic and, to a lesser extent, graphical (in some textbooks with the support of technological resources). Finally, each chapter ends with a proposal of applied and non-applied problems for the user to solve.

Table 3. Presence of PFs in the 13 textbooks analyzed (Galindo Illanes & Breda, 2023, p. 288)

PF	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	T12	T13
A	PFs on tangents												
A1	x	x	x		x	x	x	x	x	x	x	x	
A2			x		x	x	x	x	x	x			
A3	x	x	x	x	x	x	x	x	x	x	x	x	x
A4	x	x	x	x	x		x	x	x	x			
B	PFs on calculating instantaneous rates of change												
B1	x	x	x	x	x	x	x	x	x	x	x		x
B2	x	x	x	x	x	x	x	x	x	x	x	x	
B3	x	x	x	x	x	x	x	x	x	x	x	x	
B4	x	x	x	x	x	x	x						
B5	x	x	x	x	x		x	x	x	x	x	x	x
C	PFs on instantaneous variance												
C1	x	x	x	x	x	x		x	x	x	x		
C2		x	x	x	x								
D	PFs on applying the derivative for calculating maxima and minima, the analysis of graphs of functions, etc.												
D1	x	x	x	x	x	x	x	x	x	x	x	x	x
D2	x	x	x	x	x	x	x		x				
D3	x	x	x	x	x	x	x	x	x	x	x		x
D4	x	x	x	x	x	x	x	x	x	x	x	x	x
D5	x	x	x	x	x		x	x	x	x	x		
D6	x	x	x	x	x	x	x	x	x	x	x		
E	PFs on calculating derivatives from derivative rules and theorems												
E1	x	x	x	x	x	x	x	x	x	x	x	x	x
E2	x	x	x			x	x	x	x	x	x	x	x
E3		x					x	x	x	x	x		
E4		x					x	x	x	x	x		x

DISCUSSION

The analysis of the thirteen textbooks reveals that the contents in most of them are organized in a similar way for the construction of the derivative object. A predominance of symbolic language is observed in the arguments, corroborating with the studies of Vargas et al. (2020), although, in some situations, it is supported by graphic language. Relevant differences between the mathematics textbooks for civil engineering and for economics and business are observed in the PFs addressed. In civil engineering textbooks, there is a preponderance of the derivative interpreted as a ratio of change (Galindo Illanes & Breda, 2023).

With regard to the PFs on tangents, all the textbooks consider the calculation of the equation of the tangent line to a curve at a point using the derivative. On the one hand, in mathematics textbooks applied to economics and business (except T4), the construction of the slope of a tangent line to a curve at a point using the approximation by secant lines is proposed. On the other hand, textbooks T3, T5, and T6 use Leibniz's triangle. Furthermore, except for T6, the calculation of slopes of tangent lines is considered in order to determine instantaneous ratios of change. With regard to the classic textbooks on the teaching calculus for engineering, except T13, the construction of the slope of a tangent line to a curve at a point using the approximation by secant lines is proposed, mostly using Leibniz's triangle and considering problem-situations involving the calculation of slopes of tangent lines to determine instantaneous rates of change, which is in line with the studies of Santi (2011).

For the PCs on calculating instantaneous rates of change, on the one hand, all mathematics textbooks applied to economics and business consider the calculation of instantaneous rates of marginal change, relative and percentage rates of change, of instantaneous speed, and population growth problems. Except for T6, they consider rates of change applied to other areas of knowledge different from those mentioned above, coinciding with the results considered in Ariza and Llinares (2009) and Zembrano et al. (2019). On the other hand, all the classic textbooks on teaching calculus for engineering consider the calculation of rates of change applied to other areas of knowledge. With the exception of T13, all the textbooks consider problems of calculating instantaneous speed and population growth problems. Finally, it is observed that only one textbook (T7) considers relative and percentage rate of change problems for economic functions, unlike the mathematics textbooks applied to business and economics.

With regard to PFs on instantaneous variance, all mathematics textbooks applied to economics and business consider instantaneous variance of real functions and, except for T1 and T6, they all address problems of relative and percentage rates of change of real functions, approaching the proposal made by Vrancken and Engler (2014). For the classic textbooks on the teaching of calculus for engineering, T8, T9, T10, and T11 consider instantaneous variance of real functions, and none deal with problems of relative and percentage rates of change of real functions, unlike most of the mathematics textbooks applied to economics and business.

On the one hand, all the textbooks consider the analysis of monotonicity and the calculation of local maxima and minimum of real functions and of problems applied in areas of knowledge other than economics, using the criteria of the first or second derivative for relative extrema. Mathematics textbooks applied to economics and business consider the analysis of monotonicity and the calculation of local maxima and minima in problems applied to economics, using the criteria of the first or second derivative for relative extrema, and the analysis of concavity of real and economic functions. However, except for T6, all of them consider the calculation of absolute extrema in a closed interval. On the other hand, in the classic calculus engineering textbooks, only T12 does not consider the calculation of local maxima and minima in problems applied to economics and uses the criterion of the first or second derivative for relative extrema. Most of them, except for T12, add the analysis of concavity of real and economic functions and the calculation of absolute extrema in a closed interval. Only T7 and T9 consider the analysis of the monotonicity of economic functions, unlike all the mathematics textbooks applied to economics and business. These aspects characterize the PFs in applying the derivative for calculating maxima and minima, the analysis of graphs of functions, etc.

Finally, with regard to the PFs on calculating derivatives from derivative rules and theorems, all the textbooks consider the calculation of higher order derivatives using derivative rules. Moreover, the textbooks on mathematics applied to economics and business, with the exception of T4 and T5, consider the calculation of implicit derivatives. However, only T2 considers Newton's method for approximating roots of polynomials and L'Hôpital's rule as an application of derivatives. As for the classic calculus textbooks for engineering, all of them consider finding implicit derivatives, only T12 does not consider L'Hôpital's rule, and T12 and T13 do not consider Newton's method for approximating roots of polynomials.

CONCLUSION

By way of conclusion, the analysis stresses the importance of the complex approach to the derivative and, especially, the richness of its varied PFs present in the textbooks. A total of 21 problem subfields were identified in the five categories of PFs previously proposed by Pino-Fan et al. (2013), although not all the textbooks analyzed use the problem subfields extensively. It is especially relevant to highlight that not all the textbooks aimed at commercial engineering students consider the PF on tangents using Leibniz's triangle, which is an important mathematical notion employed in that area of knowledge. Furthermore, it is important to bear in mind the relative and percentage rates of change for functions of different specializations in mathematics textbooks for civil engineers.

It is considered that the information provided in this research has established a wide variety of important PFs for the design of a specific didactic proposal for learning the construction of the meaning of the derivative in engineering courses, expanding on what was presented in Galindo Illanes et al. (2023). Depending on the context, the use of technology as a didactic resource, the increasing order of difficulty in problem-situations, and the implementation of different representations are considered (Breda et al., 2021).

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REFERENCES

Aguilar, A., Valapai, H., Gallegos, H., Cerón, M., & Reyes, R. (2010). *Cálculo diferencial e integral* [Differential and integral calculus]. Pearson.

Alvarado, H., & Batanero, C. (2008). Significado del teorema central del límite en textos universitarios de probabilidad y estadística [Meaning of the central limit theorem in university texts on probability and statistics]. *Estudios Pedagógicos, 34*(2), 7-28. https://doi.org/10.4067/S0718-07052008000200001

Ariza, A., & Llinares, S. (2009). Sobre la aplicación y uso del concepto de derivada en el estudio de conceptos económico en estudiantes de bachillerato y universidad [On the application and use of the derivative concept in the study of economic concepts in high school and university students]. *Enseñanza de las Ciencias, 27*(1), 121-136. https://doi.org/10.5565/rev/ensciencias.3667

Arya, J., & Lardner, R. (2009). *Matemáticas aplicadas a la administración y la economía* [Mathematics applied to administration and economics]. Pearson.

Ballard, C., & Johnson, M. (2004). Basic math skills and performance in an introductory economics class. *Journal of Economic Education, 35*(1), 3-23. https://doi.org/10.3200/JECE.35.1.3-23

Breda, A., Pochulu, M., Sánchez, A., & Font, V. (2021). Simulation of teacher interventions in a training course of mathematics teacher educators. *Mathematics, 9*(24), Article 3228. https://doi.org/10.3390/math9243228

Budnick, F. (2007). *Matemáticas aplicadas para administración, economía y ciencias sociales* [Applied mathematics for administration, economics and social sciences]. McGraw Hill.

Butler, J., Finegan, T., & Siegfried, J. (1998). Does more calculus improve student learning intermediate micri-macroeconomic theory? *Journal of Applied Econometric, 13*(2), 185-202. https://doi.org/10.1002/(SICI)1099-1255(199803/04)13:2%3C185::AID-JAE478%3E3.0.CO;2-1

Dowling, E. (1992). *Teoría y problemas de cálculo para administración, economía y ciencias sociales* [Theory and problems of calculus for administration, economics and social sciences]. McGraw Hill.

Font, V., Godino, J. D., & Gallardo, J. (2013). The emergence of objects from mathematical practices. *Educational Studies in Mathematics, 82*, 97-124. https://doi.org/10.1007/s10649-012-9411-0

Galindo Illanes, M. K., & Breda, A. (2023). Significados de la derivada en los libros de texto de las carreras de Ingeniería Comercial en Chile [Meanings of the derivative in the textbooks of the Commercial Engineering courses in Chile]. *Bolema: Boletim de Educação Matemática, 37*(75), 271-295. https://doi.org/10.1590/1980-4415v37n75a13

Galindo Illanes, M. K., & Breda, A. (2024). Processo de instrução do derivado aplicado a estudantes de engenharia de negócios no Chile [Derivative instruction process applied to business engineering students in Chile]. *Uniciencia, 38*(1), 303-325. https://doi.org/10.15359/ru.38-1.17

Galindo Illanes, M. K., Breda, A., Chamorro Manríquez, D. D., & Alvarado Martínez, H. A. (2022). Analysis of a teaching learning process of the derivative with the use of ICT oriented to engineering students in Chile. *Eurasia Journal of Mathematics, Science and Technology Education, 18*(7), Article em2130. https://doi.org/10.29333/ejmste/12162

Galindo Illanes, M. K., Breda, A., & Alvarado Martinez, H. (2023). Diseño de un proceso de enseñanza de la derivada para estudiantes de Ingeniería Comercial en Chile [Design of a teaching process of the derivative for students of Commercial Engineering in Chile]. *Paradigma, 44*(2), 321-350. https://doi.org/10.37618/PARADIGMA.1011-2251.2023.p321-350.id1386

García, L., Azcárate, C., & Moreno, M. (2006). Creencias, concepciones y conocimiento profesional de profesores que enseñan cálculo diferencial a estudiantes de ciencias económicas [Beliefs, conceptions and professional knowledge of teachers who teach differential calculus to students of economic sciences]. *Revista Latinoamericana de Investigación en Matemática Educativa, 9*(1), 85-116.

Godino, J. D., Batanero, C., & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM–The International Journal on Mathematics Education, 39*(1-2), 127-135. https://doi.org/10.1007/s11858-006-0004-1

Godino, J. D., Batanero, C., & Font, V. (2019). The onto-semiotic approach: Implications for the prescriptive character of didactics. *For the Learning of Mathematics, 39*(1), 37-42.

Haeussler, E., Paul, R., & Wood, R. (2008). *Matemáticas para administración y la economía* [Mathematics for management and economics]. Pearson.

Hey, J. (2005). I teach economics, not algebra and calculus. *Journal of Economic Education, 36*(3), 292-304. https://doi.org/10.3200/JECE.36.3.292-304

Hoffmann, L., Bradley, G., & Rosen, K. (2006). *Cálculo para administración, economía, ciencias biológicas y sociales* [Calculus for administration, economics, biological and social sciences]. McGraw Hill.

Inglada, N., & Font, V. (2003). Significados institucionales y personales de la derivada. Conflictos semióticos relacionados con la notación incremental [Institutional and personal meanings of the derivative. Semiotic conflicts related to incremental notation]. In *Proceedings of the 19th Conference of the Interuniversity Seminar on Research in Mathematics Didactics* (pp. 1-18).

Larios, V., & Jiménez, A. R. (2022). Significados parciales de la derivada en libros universitarios en la formación de ingenieros [Partial meanings of the derivative in university books in the training of engineers]. *Praxis & Saber, 13*(33), Article e12274. https://doi.org/10.19053/22160159.v13.n33.2022.12274

Larios, V., Páez, R. E., & Moreno, H. (2021). Significados sobre la derivada evidenciados por alumnos de carreras de ingeniería en una universidad Mexicana [Meanings about the derivative evidenced by engineering students at a Mexican university]. *Avances de Investigación en Educación Matemática, 20*, 105-124. https://doi.org/10.35763/aiem20.4002

Larson, R., & Edwards, B. (2016). *Cálculo tomo 1* [Calculus volume 1]. McGraw Hill.

Leithold, L. (1998). *El cálculo* [The calculus]. Oxford University Press.

Ortiz, F. (2014). *Cálculo diferencial* [Differential calculus]. Grupo Editorial Patria.

Pino-Fan, L., Castro, W. F., Godino, J. D., & Font, V. (2013). Idoneidad epistémica del significado de la derivada en el currículo de bachillerato [Epistemic suitability of the meaning of the derivative in the high school curriculum]. *Paradigma, 34*(2), 123-150.

Pino-Fan, L., Godino, J. D., & Font, V. (2011). Faceta epistémica del conocimiento didáctico matemático sobre la derivada [Epistemic facet of mathematical didactic knowledge about the derivative]. *Educação Matemática Pesquisa Revista do Programa de Estudos Pós-Graduados em Educação Matemática, 13*(1), 141-178.

Pino-Fan, L., Godino, J. D., & Font, V. (2018). Assessing key epistemic features of didactic-mathematical knowledge of prospective teachers: The case of the derivative. *Journal of Mathematics Teacher Education, 21*(1), 63-94. https://doi.org/10.1007/s10857-016-9349-8

Purcell, E., Rigdon, S., & Varberg, D. (2007). *Cálculo diferencial e integral* [Differential and integral calculus]. Pearson.

Rodríguez-Nieto, C. A., Rodríguez-Vásquez, F. M., & Font, V. (2023). Uso combinado de la teoría extendida de las conexiones y el enfoque ontosemiótico para analizar conexiones matemáticas relacionando las gráficas de f y f′ [Combined use of the extended theory of connections and the onto-semiotic approach to analyze mathematical connections relating the graphs of f and f′]. *Estudios Educativos en Matemáticas, 114*(1), 63-88. https://doi.org/10.1007/s10649-023-10246-9

Rodríguez-Nieto, C., Rodríguez-Vásquez, F., & Font, V. (2022). Nueva mirada para analizar las conexiones desde dos lentes teóricos: La teoría ampliada de las conexiones matemáticas y el enfoque ontosemiótico [A new approach to analyzing connections from two theoretical lenses: The expanded theory of mathematical connections and the onto-semiotic approach]. In J. G. Lugo-Armenta, L. R. Pino-Fan, M. Pochulu, & W. F. Castro (Eds.), *Enfoque onto-semiótico del conocimiento y la instrucción matemáticos: Investigaciones y desarrollos en América Latin*a (pp. 193-219). Editorial Universidad de Los Lagos.

Santi, A. (2011). Objectification and semiotic function. *Educational Studies in Mathematics,* 77(2-3), 285-311. https://doi.org/10.1007/s10649-010-9296-8

Stewart, J. (2018). *Cálculo: Trascendentes tempranas* [Claculus: Early transcendental calculus]. Cengage Learning.

Tan, S. T. (2005). *Matemáticas para administración y economía* [Mathematics for administration and economics]. Thomson.

Thomas, G., Weir, M., Hass, J., & Heil, C. (2015). *Thomas' cálculo una variable* [Thomas' one variable calculus]. Pearson.

Vargas, M. F., Fernández-Plaza, J. A., & Hidalgo, J. F. R. (2020). La derivada en los libros de texto de 1º de bachillerato: Un análisis a las tareas propuestas [The derivative in the 1st year high school textbooks: An analysis of the proposed tasks]. *Avances de Investigación en Educación Matemática,* (18), 87-102. https://doi.org/10.35763/aiem.v0i18.288

Vrancken, S., & Engler, A. (2014). Una introducción a la derivada desde la variación y el cambio: Resultados de una investigación con estudiantes de primer año de la universidad [An introduction to the derivative from variation and change: Results of a research with first-year university students]. *Bolema: Boletim de Educação Matemática, 28*(48), 449-468. https://doi.org/10.1590/1980-4415v28n48a22

Zambrano, R., Escudero, D. y Medrano, R. (2019). Una introducción al concepto de derivada en estudiantes de bachillerato a través del análisis de situaciones de variación, una introducción al concepto de derivada en estudiantes de secundaria [An introduction to the concept of derivative in high school students through the analysis of variation situations, an introduction to the concept of derivative in secondary school students]. *Educación Matemática, 31*(1), 258-280. https://doi.org/10.24844/EM3101.10