MODESTUM EURASIA Journal of Mathematics, Science and Technology Education, 2024, 20(12), em2542
ISSN:1305-8223 (online)
OPEN ACCESS Research Paper https://doi.org/10.29333/ejmste/15656

Analysis of primary school students' process of understanding about the concept of ratio: A view from the Pirie-Kieren theory

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Received 24 February 2024 ▪ Accepted 05 November 2024

Abstract
The understanding process of primary education students was analyzed when they solve tasks related to the concept of ratio. The study was based on the theoretical framework of Pirie and Kieren (1994). The methodology was qualitative with the case study method. The study was carried out in three stages: planning, development and analysis, using the field observation technique. Data collection was carried out through a task and an interview. The data were analyzed based on theoretical articulation. The results revealed that students lack the prior knowledge necessary to understand the concept of ratio. In conclusion, it can be noted that students do not present logical arguments to formalize the concept, and their understanding process is reduced to memorization or the use of mathematical strategies without understanding the relationship between the task and the mathematical concept.

Keywords: mathematical understanding, ratio, fraction, difficulties, previous knowledge

INTRODUCTION

Understanding mathematical concepts is essential in the learning process of every person. From school to university education, students must be able to establish connections between different mathematical concepts to be able to apply them in their daily lives and solve specific problems (Patmaniar et al., 2021). The National Council of Teachers of Mathematics (NCTM, 2000) has established that understanding is essential for the active construction of new knowledge from students' experiences and prior knowledge. As explained by Rodríguez-Nieto et al. (2023, 2024), when a person uses mathematics, they are able to establish mathematical connections that involve meanings, properties, representations, strategies, arguments, metaphors, among others. That is why the objective of the mathematics teaching process is to guarantee that students learn each of the topics from understanding (Hiebert & Carpenter, 1992; Kastberg, 2002).

Sierpinska (1990) stated that there are four important acts for understanding mathematical concepts. The first is found in the identification of objects, based on their discovery or recognition. Second, in discrimination to recognize differences between two or more objects. The third, in appropriating the objects to generalize them; and finally establish an organized and unified synthesis of several generalizations, thus formalizing the understanding of a mathematical topic. Manifestations such as these (e.g., Albert & Kim, 2015; Arnon et al., 2014; Kastberg, 2002; National Research Council [NRC], 2001; Pirie & Kieren, 1994; Skemp, 1980), have made important contributions to the study of mathematical understanding. However, it is relevant to continue carrying out research that has an impact on the mathematics teaching process and that leads to greater educational quality.

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Contribution to the literature

- This article analyzes the comprehension process that primary school students experience when addressing tasks related to the concept of ratio in complex situations.
- The difficulties that these primary school students face when trying to understand the concept of ratio in complex situations are identified.
- Teaching strategies and approaches aimed at improving students' mathematical understanding are proposed.

In fact, many students' difficulties are still perceived in solving mathematical problems, such as exponential problems (Syafiqoh et al., 2018), real function problems (Rodríguez-Vásquez & Arenas-Peñaloza, 2021), statistical problems (García-García et al., 2020), among others. Furthermore, the research by Arenas-Peñaloza and Rodríguez-Vásquez (2022), concluded that students are unable to formalize their understanding process in relation to the concept of ratio, due to the difficulties they present in correctly applying mathematical strategies when solving tasks proposals.

In relation to the concept of ratio, it is understood as that abstract number that expresses the relationship between two magnitudes of the same species or is the number that results from comparing two magnitudes of the same species by quotient. In general, if a and b are quantities of the same magnitude, their ratio is the indicated quotient or quotient that results from dividing the measure of a by the measure of b, is called ratio between a and b, and it is written $\frac{a}{b}$ or $a:b$ (Caballero et al., 1970). Therefore, it plays an important role in the teaching-learning process of mathematics, due to its connection to the understanding of other mathematical topics such as measurement, proportionality, among others. Furthermore, a good understanding of the topic allows improving the learning of other disciplines such as physics and geography (Heinz & Sterba-Boatwright, 2008). Despite this, it has been found that during the process of learning the concept of ratio, students manifest multiple difficulties. For example, Wahyuningrum et a. (2023) identified that students have not been able to identify the topic in a mathematical task, sometimes they recognize the multiplicative relationship between quantities, but they do not relate it to the concept of ratio. Other research has shown that students use fractions to represent a rate but fail to establish the relationship between fraction and ratio (Andini & Al Jupri, 2017; Wahyuningrum et al., 2017).

Fauziah and Cahya (2021) state that students have three difficulties in common when solving problems related to the concept of ratio. The first, the lack of reading understanding to interpret and solve verbal problems that imply a ratio; the second, aimed at the difficulty in correctly applying the procedures with arithmetic operations and the last, that they fail to understand the concept of ratio.

Furthermore, they mention that these difficulties are due to the lack of understanding they have about basic and arithmetic knowledge, such as the concept of fraction, relationship, measurement, among others. Similarly, Wahyuningrum et al., (2023) revealed that the essential prior knowledge for teaching and learning ratio must be correctly activated in students, in order to know their ways of thinking and understand the concept itself. Since students' prior knowledge should help them learn and not hinder their learning. All these difficulties affect the process of students' understanding of the concept of ratio.

After reviewing the literature, the importance of studying mathematical understanding from different perspectives has been recognized and it has been identified that the concept of ratio is relevant to the study of mathematics and other subjects. However, the literature in the field of Mathematics Education has reported that students have not been able to formalize their understanding process in relation to the concept of ratio due to the difficulties they present when applying mathematical resolution strategies (Arenas-Peñaloza & Rodríguez-Vásquez, 2022). Therefore, it is necessary to know the errors made by students to identify the causes and thus establish solutions to improve the quality of mathematics learning. In this sense, the objective of this study is to analyze the understanding process of primary education students when solving tasks related to the concept of ratio, using the levels of understanding proposed by Pirie and Kieren (1994) as a theoretical framework.

THEORETICAL FRAMEWORK

The mathematical understanding model of Pirie and Kieren (1994) was used, which describes a person's understanding process in a dynamic, recursive and non-linear way. It recognizes understanding as a continuous process that is built iteratively from the experiences of the subject to specify an object and thus build, strengthen or modify knowledge of it. The model is structured in eight levels that describe the process of students' understanding of a mathematical concept (see **Figure 1**).

At these levels, progress can be made by advancing or going back to a previous level with the aim of reflecting or reworking on previous understandings (Arenas-Peñaloza & Rodríguez-Vásquez, 2022) which

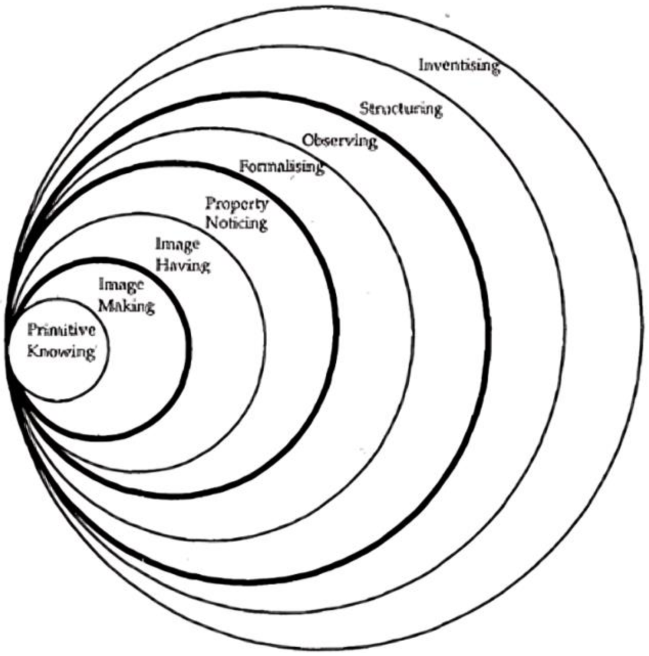

Figure 1. Levels of the understanding process according to Pirie and Kieren (1994)

the authors call folding back backward). The eight levels of the model are:

Primitive knowledge (PK): This level refers to the knowledge and skills that a student has before confronting a new mathematical concept. It is important to note that "PK" does not imply a low level of mathematical knowledge, but rather the starting point for the growth of any particular mathematical understanding. Pirie and Kieren (1994) emphasize that this level is not a reflection of the student's ability or intelligence, but rather a necessary basis for building new mathematical knowledge.

Image creation (IC): At this level, students are able to make distinctions based on their abilities and prior knowledge and perform physical or mental actions to create an idea of the new mathematical concept. It is important to note that images are not always pictorial representations but can also be expressed through language or specific actions of students. This level is essential for the development of a deeper and more solid understanding of mathematical concepts.

Having the image (HI): At this level, students are able to use a mental construction about the mathematical concept, but without the need to work with particular examples or make an abstraction of the same concept. The student finds it necessary to replace the images associated with the concept with a mental image of it. Therefore, it is achieved when a representation (symbolic, pictorial, graphic, among others) of the situation associated with the mathematical object is established.

Property observation (PO): At this level, the student demonstrates logical thinking skills by using and combining aspects of the mental images they already possess. This allows you to construct specific properties of the concept and try to generalize them. Therefore, she is capable of thinking orderly, reasoning, analyzing, comparing and synthesizing information.

Formalization (F): At the fifth level, the student is able to work on the mathematical concept as a formal object and does not refer to a particular action or image. To demonstrate that the student knows the properties and is able to abstract common characteristics of an image, the student's ability to identify patterns and relationships between mathematical elements can be observed. For example, if a series of geometric figures with different sizes and shapes is presented, the student must be able to identify the common properties of all the figures, such as the number of sides, the presence of angles, among others. Additionally, she must be able to use mathematical language to describe these properties and relationships precisely and formally.

Observation (O): It refers to the student's thinking ability to use their formal mathematical language, reflecting on formal statements and making connections between mathematical concepts. This allows you to deduce patterns and regularities when expressing algorithms and theorems. In addition to communicating precise and clear mathematical ideas, and for solving mathematical problems.

Structuring (S): At this level, the student is expected to reflect on the Os made and consider them formally as a theory. Also, justify or verify the statements through a logical or meta-mathematical argument.

Invention (I): At this last level, the student is able to fully understand a concept and detach himself from specific and determined situations to undertake other perspectives that lead him to make hypotheses about another problem or concept. This implies that the student has reached a level of abstraction in their mathematical understanding. By reaching this level, the student can apply the concept in different situations and solve more complex problems. Additionally, she can formulate mathematical hypotheses and theories based on a complete understanding of the concept. To reach this level of understanding, it is important that the student has a solid foundation in previous mathematical concepts and has adequately navigated the previous levels.

MATERIALS AND METHODS

This research is qualitative descriptive (Cohen et al., 2018) and focuses on analyzing the comprehension process of primary education students when solving tasks related to the concept of ratio. The study will be carried out in four stages: the first, the participants were selected; The second corresponds to the design of the task in which they had to use comparison strategies between mathematical ratio to choose the best option; the third, to the planning and development of the sessions worked on and, the last, consisted of the analysis of the mathematical activity of the students during the process of understanding the concept of ratio. The participants in the study were four students between 11 and 12 years old who attended sixth grade in a Mexican primary school. They were selected because they had already developed the characterization of the concept of ratio. Two of the students had high academic performance (case 1), while the other two had low academic performance (case 2). To identify the students, they are credited with codes: S1 and S2 for those in case 1, and S3 and S4 for those in case 2.

To collect the data, a mathematical task related to the concept of ratio was obtained (**Figure 2**). Subsequently, semi-structured interviews were applied to the students to deepen their way of proceeding during the activity carried out (Cohen et al., 2018). The task was developed in two sessions, one for each case study, which were video-recorded, and field notes were collected. Finally, the data were transcribed and compared, interpreted and analyzed in relation to the characteristics and levels of action of the theoretical model of Pirie and Kieren (1994).

The Task

The task called "share, enjoy and learn" was proposed to analyze students' understanding of the concept of ratio in a context outside the classroom, specifically in a pizza restaurant. The objective of the task (**Figure 2**) is for students to apply comparison strategies between mathematical ratios to choose the best option. The materials and resources they used to solve the task were sheets of paper, graphite pencil, eraser, and pencil sharpener.

Every month Maria and her friends meet at the Alamedilla restaurant for pizza dinner. María usually arrives late, but her friends appreciate her very much and wait for her. They reserve a place for him at each of the two tables they occupy. On table 1 there are 4 large pizzas (non-cut) and 5 people, on table 2 there are 6 large pizzas (non-cut) and 7 people.

a) Graphically represent the situation at each table before María arrives.

Maria arrives with a lot of appetite (hungry) and has to decide where to sit so that she gets the largest amount of pizza.

b) Graphically represent the situation that results if María decides to sit at table 1.
c) How much pizza does each of them get, if María chose to sit at table 1? Justify your answer.
d) Now, graphically represent the situation that results if María decides to sit at table 2.
e) That being so, how much pizza does each of them get, if María chose to sit at table 2? Justify her answer.
f) Which of the two tables do you suggest Maria choose to sit at? Explain her reasoning.
g) Suppose that María selected table 1, in which they add 2 pizzas to the 4 they already had. To maintain the same distribution that they had previously, what will be the number of people that should be added? Explain her reasoning.
h) If María chose table 2, 4 more friends arrived and they want to eat the same amount of pizza as before. How many more pizzas should they buy? Explain your answer.
i) If you want to place 240 people at large tables (for 8 people) and small tables (for 6 people), maintaining a ratio of 7 large tables to 4 small tables, how many tables of each type are needed?

Figure 2. Task "share, enjoy and learn" (Source: Authors' own elaboration)

The task in **Figure 2** was structured into nine questions based on Pirie and Kieren's (1994) model of mathematical understanding. (a) requires students to graphically represent the two situations at each table before María chooses where to sit, which aims for the student to have a general idea and deduce from that what would happen if María decided on one of the tables. Also, it was designed to identify the difficulty in using standard figures that students present according to the literature report. In (b) they are asked to graphically represent the situation at table 1 but with the condition that María has chosen said table. The aim is for students to use their reasoning and compare the two situations through the representation they made in the previous exercise (question a). In question (c), students are asked to establish the mathematical ratio that results from the situation at table 1 if Maria is chosen, in relation to the amount of pizza times the number of people. For its part, in question (d) and question (e), the same questions are raised as in exercise (b) and exercise (c), respectively, but in relation to the situation that results from table 2 if María decides to sit at They seek for students, through representations, to establish relationships between the quantities that occur in each of the situations presented.

In question (f), students are expected to ratio and argue which is the best option so that Maria can sit down and eat more pizza. They are expected to make this determination through previously constructed representations or using some ratio comparison strategy. Question (g) and question (h) require students to demonstrate a deeper understanding of the notion of ratio, which involves higher-level proportional thinking. In question (g), students must recognize that, if 2 pizzas are added to table 1, 3 people must be added to maintain the same initial proportion $\left(\frac{2}{3}\right)$. Similarly, in question (h), the reverse situation arises, where when 4 more people arrive at table 2, 3 more pizzas must be added to maintain the same distribution that they had previously $\left(\frac{3}{4}\right)$.

Question (i) asks students to interpret, reason and argue the situation presented, using distribution strategies to determine the number of people who should sit at each of the tables (large and small), maintaining a proportion of 7 tables. large to 4 small. This requires students to apply mathematical concepts and ratio critically to arrive at an appropriate solution. According to the mathematical understanding model of Pirie and Kieren (1994), questions (a), (b), and (d) were designed for the first three levels, while questions (c) and (e) correspond to the fourth level and question (f) at the fifth level. On the other hand, questions (g), (h), and (i) are designed thinking about the higher levels of the model (sixth, seventh, and eighth level), seeking that students will develop their ability to establish and apply mathematical ratios in contexts practical, promoting more advanced proportional thinking.

Data Analysis

The theoretical model of Pirie and Kieren (1994) was applied to analyze the data. Based on field O, the specific characteristics of some levels of understanding were identified and described, in relation to the problem situation (**Figure 2**).

At the first level of PK, we sought to identify and describe the prior knowledge of the cases studied, related to equitable distribution, graphic representation, order of fractions and strategies for comparing mathematical ratios. In the second level of IC, the distinctions based on their prior knowledge were analyzed in each case, through the relationship between the number of pizzas and the number of people at each table. At the third level, HI, it was identified whether the cases were able to establish a representation (symbolic, pictorial, graphic, among others) of the distribution situation of the tables associated with the concept of ratio (relationship between the number of pizzas and people). At the fourth level of PO, the abilities in the use of strategies that would allow a mathematical ratio (presented as a fraction) to be compared in a favorable manner were determined. At the fifth F level, we sought to identify the moment in which the cases managed to correctly recognize the best option that María had to sit down and eat the largest amount of pizza (table 2), which allowed them to use their prior knowledge to establish the representation of the situation related to the concept of ratio and detaching from it. In this way, they have managed to use mathematical properties to formalize the concept of ratio.

In relation to subsequent levels of understanding, we sought to identify the ability of students to correctly establish and apply mathematical ratio in practical contexts, where they needed to use the already formalized concept to promote more advanced proportional thinking.

RESULTS

The results are presented in two sections, one for each case study, considering the development of the task by the students (process of understanding case 1 and case 2). These were divided into three differentiated categories according to the students' responses and the objective of the task: choosing the best ratio, identifying the ratio and distribution strategies.

Case 1's Understanding Process

The analysis of the process of the knowledge structures of the students of case 1 (S1 and S2) is presented, based on the theoretical model of Pirie and Kieren (1994) when solving the task (**Figure 2**).

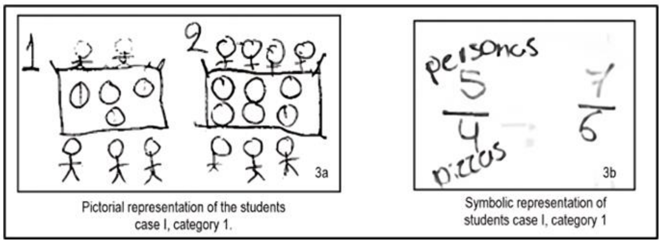

Figure 3. Representations of the students case 1, category 1 (Source: Authors' own elaboration)

Choosing the best ratio (category 1)

The pair of students (case 1) seemed calm and confident in solving the task, they organized themselves as a team and it was S2 who read the statement aloud. The first thing the students do is analyze in detail the two situations (table 1 and table 2) that the task provides them with.

> S1: You read.
>
> S2: Ok, I read… [starts reading].
>
> S1: Do we have to do the fraction?
>
> S2: We have to represent graphically.

The students began solving the task by relating each situation on the tables with a graphic representation, which indicates that, according to the Pirie and Kieren's (1994) model, they began the task from the image making level by relating the mathematical object with the creation of a visual representation. of the two situations (see part 3a in **Figure 3** and excerpt from the transcript).

> S1: Like this [they draw two rectangles that represent the tables for them].
>
> S2: Yes. Here are four pizzas [points to the representation of table 1].
>
> S1: Including María?
>
> S2: No, without María.

Subsequently, the students made a pictorial representation of the situation presented at both tables (see part 3a in **Figure 3**). According to Pirie and Kieren's (1994) model, this indicates that the students advanced to the image taking level, where they continued to make another type of representation, in this case a symbolic representation of the two situations (see part 3b in **Figure 3**). It is important to highlight that these first representations that the students made were made considering that María had not selected either of the two tables yet (see following excerpt from the transcript).

> S1: You have to do the fractions.

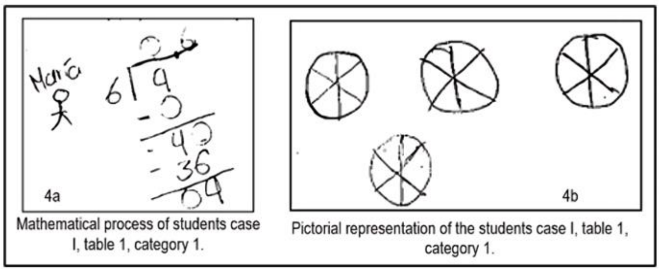

Figure 4. Student representations of case 1, table 1, category 1 (Source: Authors' own elaboration)

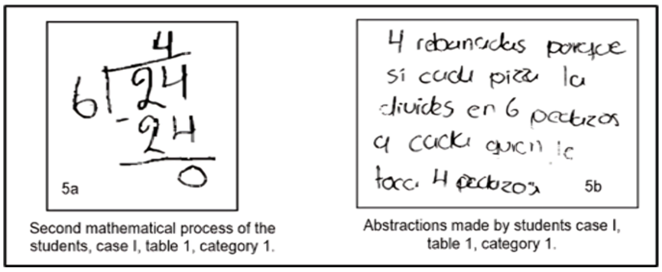

Figure 5. Second mathematical processes of students and abstractions (Source: Authors' own elaboration)

S1: Perform the fractions [hands the pencil to S2].

S2: Five quarters? [refers to the situation on table 1].

S1: Yes.

In part 3b in Figure 3, it is observed that students face difficulties in the mathematical ratio relationship. Instead of distributing people (denominator) in relation to the number of pizzas (numerator) $\frac{4}{5}$ and $\frac{6}{7}$, they related the representations in an opposite way.

S1: How many people would it be?

S2: Here there would be six with her [number of people at table 1 adding María].

S1: We have to … operate. To know where it is best for us to eat more pizza.

S2: Yes, where is it convenient for us to eat more pizza.

Subsequently, the students discovered that, by having a symbolic representation of the two situations, they needed to attach Mary to each of them and divide each of the pizzas in such a way that each person would receive the same amount of pizza. Their objective was to determine which of the two tables was most convenient for María. This process shows that the students were working with the already established representation and had advanced to the property noticing level. While solving the problem, the students performed the algorithm of dividing 6 by 4, including María at table 1. However, they realized that the quotient was 1.5, which implied that each person would receive a pizza and average, and they only have four pizzas for six people. Then, he understood that the ratio had to be done the other way around, 4 is 6 (including Mary at table 1). The students erased the previous algorithmic process and performed a new division (part 4a in Figure 4 and transcript excerpt).

S1: Let's divide like this [performs the arithmetic operation of dividing 4 by 6].

S2: Yes. Four out of six.

Then, having the symbolic representation of the two situations, the students noticed that they had to add Maria to each one and divide the pizzas so that each person received an equitable amount. However, they stuck to arithmetic processes instead of reasoning about the equitable distribution of the pizzas. Subsequently, they created another representation of table 1 based on the result obtained, dividing the four pizzas into six equal parts (see part 4b in Figure 4 and excerpt from the transcript). Although this could be interpreted as a retreat to a lower level, it is perceived that the students returned with a more solid understanding and associated this representation with the abstractions made, now finding themselves at the image taking level.

S2: Each pizza would have to be divided into six [divide each pizza into six parts].

S1: Shall we do all four? [refers to representing the four pizzas].

S2: Yes, four.

The students determined the total number of portions obtained by dividing each pizza into six equal parts. Then, they calculated how many portions of pizza each person at table 1 would get if Maria decided to sit there, using the arithmetic operation of division (part 5a in Figure 5). This process indicates that the students found themselves again at the property noticing level, since they obtained information from the representation of it and worked on it (see excerpt from the transcript).

S1: Six times four, twenty. Shall we divide twenty among six people? [the student performs this operation by observing the representation they built of table 1 already divided].

S2: No, six times four is twenty-four.

S1: We divide twenty-four by six, right? [refers to the total number of pizza portions among the total number of people at table 1].

S2: Yes, between six.

The students were able to synthesize the information from the arithmetic process carried out for table 1,

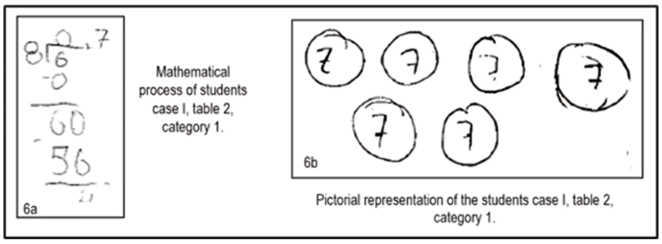

Figure 6. Mathematical process of the students of case 1 and pictorial representation (Source: Authors' own elaboration)

concluding that, by dividing each pizza into 6 equal parts, each person receives 4 portions (see part 5b in **Figure 5** and extract from the transcription).

S1: Does the pizza have six portions?

S2: It's not like that, they would have four portions each.

S1: Then it would be this [points to the representation of table 1].

S2: Yes. At table 1 each person would get four portions.

After understanding the situation at table 1, students apply the same process to table 2, dividing 6 pizzas among 8 people, including Maria, to determine how to divide each pizza (see part 6a in **Figure 6**). There is evidence that they advance to the property noticing level. Then, they pictorially represented the 6 pizzas divided into seven equal parts, although they should have been divided into eight equal parts (see part 6b in **Figure 6** and excerpt from the transcript). This indicates that they have performed a folding back but with a more solid understanding, since they associated this representation with the abstractions made and returned to the image having level.

S2: The same as we did with table 1.

S1: Yes, people have to be divided...

S2: No, first the pizzas are divided among the eight people.

S1: But aren't there seven people?

S2: Between eight, because if María sits at table 2.

S1: The pizzas are divided into seven parts [refers to dividing them into seven equal parts]. You have to make the pizzas [refers to representing them].

Students begin to obtain information about the total number of portions they would have if they divided each pizza on table 2 into seven equal parts. Then, they seek to determine how many portions of pizza will correspond to each person at table 2 if Maria decides to sit at it, using the process of dividing 42 portions among

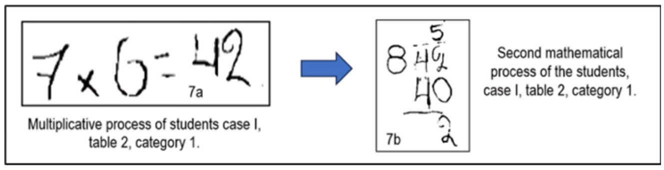

Figure 7. Students' multiplicative process and second mathematical process case 1 (Source: Authors' own elaboration)

Figure 8. Abstractions made by the students of case 1 (Source: Authors' own elaboration)

the 8 people (see part 7a in **Figure 7** and transcript excerpt). This shows that students return to the property noticing level, using a comprehension process similar to that of the situation in table 1.

S2: Then seven times six equals forty-two [refers to the total number of portions obtained by dividing each of the six pizzas into seven parts].

S1: Yes [while performing the operation of multiplying seven by six].

S2: We divide that by eight (see part 7b in **Figure 7**).

It is observed that the students have abstracted in

formation from the division process and have determined how many portions of pizza correspond to each person at table 2 if María decides to sit at it (see part 8a in **Figure 8** and extract from the transcript).

S1: Then each one will take five servings.

S2: Yes, but we have to see if there are five.

S1: Yes, because five times eight is forty ... we have forty-two portions, therefore, there are two left over.

S1: So, each one gets five portions and there are two left over.

The students responded to the question of which table they would recommend Maria to sit at, concluding that table 2 is more suitable for her, since she could eat a greater number of pizzas there (see part 8b in **Figure 8**). Even if they select the best option, they will show difficulties in the process of comparing the reasons that trigger both situations (table 1 and table 2). This indicates that the students did not reach the Formalizing level.

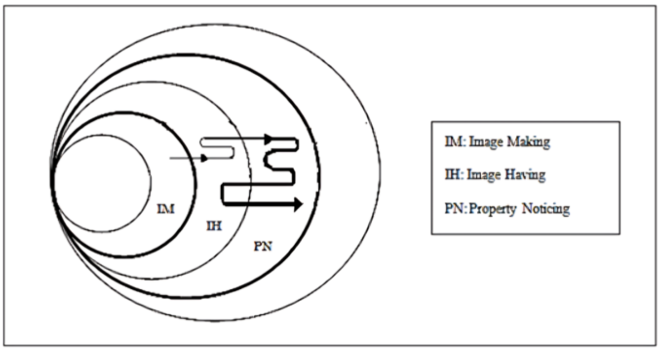

Figure 9. Students' knowledge structure to choose the best ratio, case 1, category 1 (Source: Authors' own elaboration)

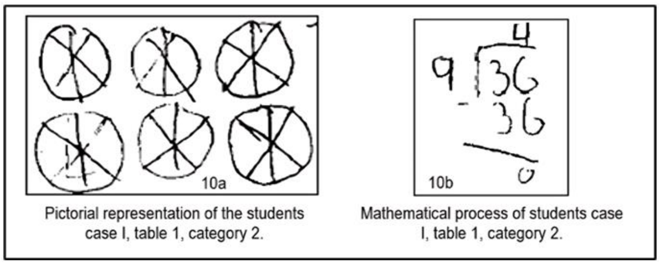

Figure 10. Pictorial representation and mathematical process of students case 1, table 1, category (Source: Authors' own elaboration)

Figure 9 schematizes the process followed by the students to determine the choice of the best mathematical ratio (referring to the table where María would eat the most pizzas). The arrows in **Figure 9** indicate that the case studied reached that level in its understanding process, and the thickness of the line represents the development of said process.

Identification of the ratio (category 2)

This category of analysis corresponds to questions (g) and (h) of the task (**Figure 2**), in which students must identify the relationship between both mathematical ratios. To achieve this, it is necessary that they reach a deeper understanding of the notion of mathematical ratio and develop more advanced proportional thinking. In relation to question (g), students begin by creating a visual representation of the new situation, adding two pizzas to the existing four, as indicated in the task (see part 10a in **Figure 10** and excerpt from the transcript). This shows that students are addressing this new question from the level of image having. Subsequently, they begin to abstract information from the image and decide to divide the two added pizzas into the same six parts as they had done previously.

> S2: Now add two pizzas to this [points to the representation they had made of table 1 already divided].

> S1: Six times six is thirty-six [refers to the six units of pizza divided into six parts].

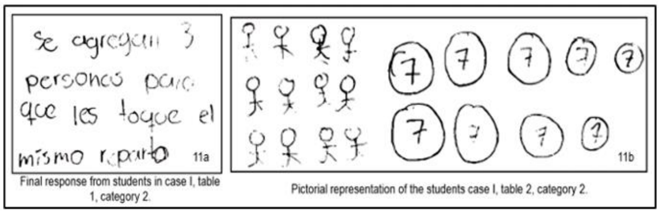

Figure 11. Final response and pictorial representations of the students case 1, table 1, category 2 (Source: Authors' own elaboration)

> S2: Three people are added. Because each person gets four servings, right? [expresses it based on the result obtained].

Students, by adding two pizzas to the existing four, immediately demonstrate an understanding of how many people should be added, although they perform the division process to confirm their deductions (see part 10b in **Figure 10** and transcript excerpt). This shows that the students are at the property noticing level, since they work with the image they have. In the interview, the students express that, with the help of representation and trial, they obtained their results.

> S1: Shall we divide? [He expresses it to verify the answer they had already deduced].

> S2: Yes.

> S1: How much is it? Thirty-six out of nine, right?

> S2: Yes.

The students finally detach themselves from the image representing the relationship of table 1 and manage to formalize their understanding in relation to the need to add three people to the six that already existed, including María, to maintain the same distribution (four portions each person) but with six pizzas (see part 11a in **Figure 11** and transcript extract). This gives evidence that the students reached the Formalizing level.

As for question (h), students decide to apply the same process used in question (g) for table 1, but this time for table 2. They begin with a visual representation of the situation on table 2, drawing the twelve people and the initial six pizzas (see part 11b in **Figure 11** and transcript excerpt). This shows that students are approaching the task from the level of image having.

> S1: You have to add your friends to know how many pizzas there are.

> S2: Not the pizzas, because you have to know how many more pizzas you have to buy.

> S1: So it would be twelve people, right?

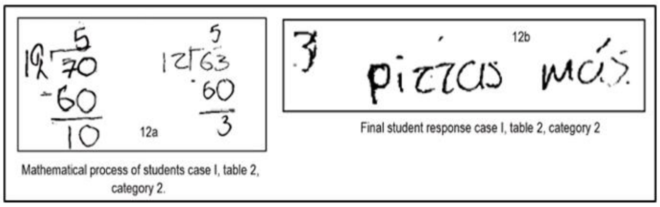

Figure 12. Mathematical process of students case 1, table 2, category 2 (Source: Authors' own elaboration)

S2: At table 2 there are seven people and with María there would be eight.

S1: Then seven plus four would be eleven, eleven plus one would be twelve [adds the number of people at table 2 with the condition of adding four and María].

S2: There would be twelve people then, because it says here that "if María chose table 2 …"

The students, when addressing question (h), maintain the same incorrect fractionation, dividing each pizza into seven equal parts (see part 11b in **Figure 11**), indicating that they have not adequately considered the division process. They first use trial and error and then secure their deductions by division. This shows that the students have advanced to the level of property noticing.

S1: It must be done as was done with the previous one [refers to the list of pizza per person on table 1, carried out previously].

S2: Yes, but how many pizzas should we add?

S1: Initially there are six pizzas [represents the six pizzas with circumferences and divides each one into seven parts].

S1: The six pizzas are divided into seven parts, there are 42 portions.

Students, working on the existing representation and adding pizzas to the initial six, initially fail to identify the relationship between the number of people and the quantity of pizza. By adding two pizzas, you perform the operation of dividing 56 portions among the 12 people, resulting in 4 portions per person, contradicting the initial distribution of 5 portions per person and leaving 2 additional portions. This shows that the students have advanced to the level of property noticing.

S1: There would be four portions.

S2: Then no because it has to come out in five servings like before.

In this sense, the students increase the number of pizzas and add 4 pizzas to the initial 6, adding a total of 70 portions to be divided among 12 people, obtaining an integer quotient of 5 and a remainder of 10. Then, they deduce that they must remove a pizza and add 3 to see how it turns out (see part 12a in **Figure 12** and transcript excerpt).

S1: You have to add four pizzas, because they make 5 portions, but there are ten left over.

S2: I think we should remove the pizza. Then it would be adding three pizzas.

S2: With these two pizzas there would be fifty-six portions [points to the representation of the pizzas when two are added].

S2: If we add another one [a total of nine pizzas] it would be sixty-three portions [each one divided into seven parts].

S2: Then there would be sixty-three portions between twelve people.

S2: There are three portions left, they would be less.

The students faced difficulties when addressing the situation of table 2, both in the first and the second category of analysis, by not correctly interpreting the ratio for table 2. In the end, the students overcome the created image and the mathematical process, making final guesses. This progress shows that the students will reach the level of formalizing, by stating that if four more people arrive to the initial eight (with María), 3 pizzas must be added to table 2 to maintain the same initial distribution (5 portions of pizza), see part 12b in **Figure 12**.

S1: Then it would be adding ...

S2: Three pizzas must be added, so that each person gets the same share.

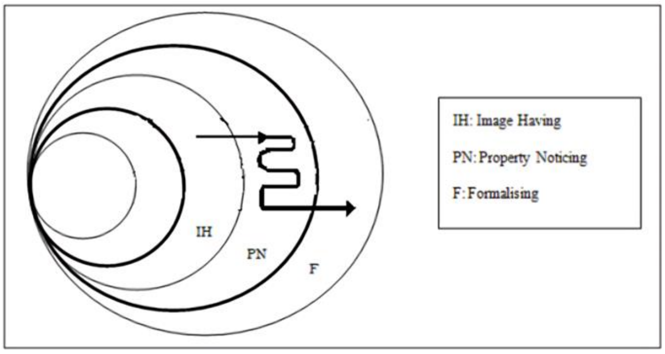

Figure 13. Students' knowledge structure to identify the ratio, case 1, category 2 (Source: Authors' own elaboration)

Figure 13 schematizes the process that the students followed to identify and determine the relationship between the number of pizzas and people on table 1 and table 2. The arrows indicate that the case under study

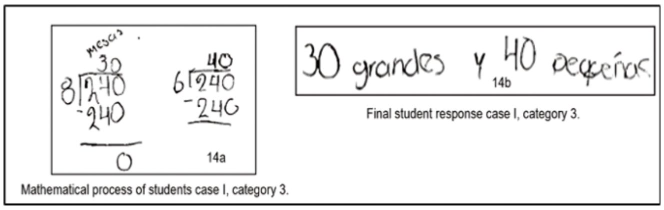

Figure 14. Mathematical processes of the students of case 1, category 3 (Source: Authors' own elaboration)

achieved that level in its understanding process, and the thickness of the line represents the development of the understanding process.

Distribution strategies (category 3)

The last category of analysis seeks to understand students' distribution strategies in complex situations that involve mathematical ratio. At this stage, students must determine the number of people who should sit at large and small tables, maintaining a ratio of 7 large tables to 4 small tables. Students show confusion when addressing the last question of the task (question i), expressing their confusion and reading the situation carefully several times to try to solve it.

> S1:Read this last question.
>
> S1: I didn't understand [expresses it when S2 finishes reading].
>
> S2: We have to place the 240 people on the large and small tables with a ratio of 7 large tables to 4 small ones.

The students began to solve the question from the Image Taking level, since they were creating mental images to determine the proportion of tables. However, they did it separately with each type of table, without using the given proportion (7 large tables to 4 small ones), see part 14a in **Figure 14** and excerpt from the transcript.

> S1: Aha, shall we divide?
>
> S1: It would be 240 divided by 8.
>
> S2: Yes. It would be 30 [refers to the situation of the large tables].
>
> S1: So what does this mean … Thirty tables? [He expresses it when he finishes the division and obtains 30 as a quotient].

The students were unable to interpret the situation of the last question of the task and began to work on the mental image they had, using division to obtain their answer (see part 14a in **Figure 14**). This shows that the students have advanced to the level of property noticing. However, they did not consider the given proportion (7 large tables to 4 small ones), which was asked in the interviews, and they stated that they did not understand what to do with that data. This shows that students do not have a good understanding of the concept of ratio in complex situations. Then, the students tried to generalize the process they used to solve the question, expressing the number of large and small tables necessary to accommodate 240 people (part 14b in **Figure 14** and extract from the transcript), but incorrectly, since they did not consider the ratio (7: 4) of the tables. The students remained at the same level (property noticing).

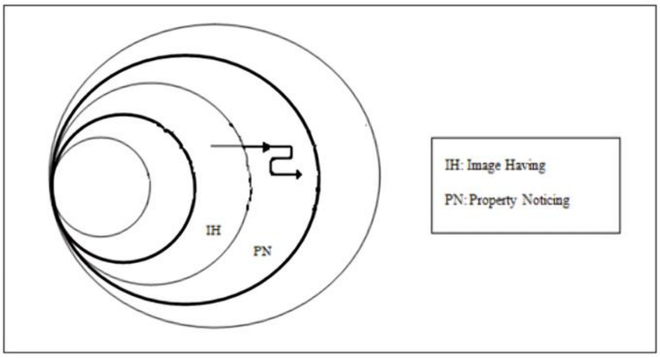

Figure 15. Students' knowledge structure using distribution strategies, case 1, category 3 (Source: Authors' own elaboration)

> S1: Then there would be 30 large tables and 40 small tables.
>
> S2: Yes [expresses it by shaking his head].

Figure 15 schematizes the process that students followed to determine distribution strategies in situations that involve mathematical ratios. The arrows indicate that the case under study achieved that level in its understanding process, and the thickness of the line represents the development of the understanding process.

Case 2's Understanding Process

The analysis of the process of the knowledge structures of the students of case 2 (S3 and S4) is presented, based on the theoretical model of Pirie and Kieren (1994) when solving the task (**Figure 2**).

Choosing the best ratio (category 1)

The pair of students (case 2) seemed distracted when they began to solve the task. S4 read the statement aloud, while both related each of the two situations (table 1 and table 2) that the task offered them with a representation. This shows that the students began to solve the task from the image having level.

> S4: You read the first question.

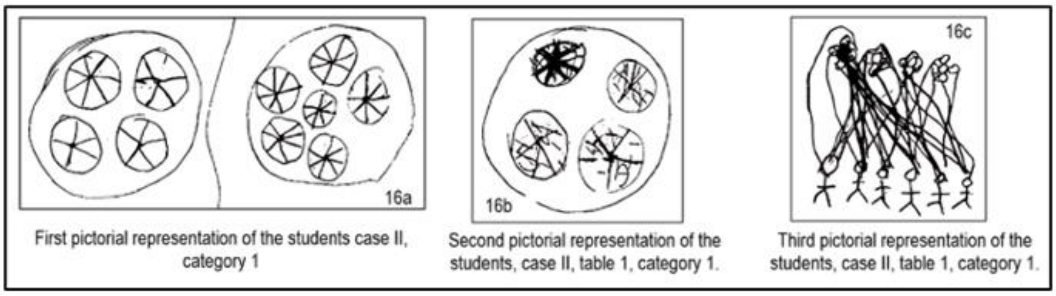

Figure 16. Pictorial representations of students case 2, category 1 (Source: Authors' own elaboration)

S3: I don't know ... I remember that last time we represented with some circles.

S4: Oh yes! but this time it's some pizzas.

S3: Yes.

After creating two pictorial representations (part 16a in **Figure 16** and transcript excerpt), students identified that each pizza could be divided according to the number of people present at each table. However, it is important to clarify that the representations made by the students of the two table situations did not contain María.

S4: Let's start, let's put some circles, how many pizzas?

S4: Now we divide them into eight [refers to dividing each pizza into eight parts].

S3: Should they be divided into five, right? [Refers to dividing each pizza into five parts].

S4: No, in seven parts [S3 divides each pizza into seven equal parts].

To determine which of the two tables was the most convenient for Maria, the students created new representations that included Maria at both tables and divided the pizzas in such a way that each person would get the same number of pizzas (part 16b in **Figure 16** and excerpt from transcription). This process demonstrates that the students remained at the image having level. The students had to try their hand at dividing each pizza and distributing them to the six people, including Maria. First, they divided it into halves, thus obtaining eight portions in total, but they realized that they had two portions left over.

S4: How many people does table 1 have?

S3: On table 1 there are four uncut pizzas and five people. So, we cut a pizza in two [refers to dividing each pizza into 1/2].

S4: Then we split them two by two.

S3: Yes. Not in three? [refers to dividing each pizza into three parts].

During the process of dividing the pizzas, the students proposed another distribution of dividing them into three equal parts. However, they were confused when they did not realize that, by dividing the four pizzas into thirds, they would obtain twelve portions, which means that, since there were six people at table 1 (including María), each one would receive exactly $\frac{2}{3}$ of the four pizzas. This episode shows the students' difficulties with some mathematical concepts and processes necessary to make an equitable distribution.

S3: I followed it, what you did first. There are six people, and each one has to get the same.

S3: Let's see there are six people ... How much do we have to divide each pizza into? In five no.

S4: Yes, five portions each pizza. Twenty portions in total? Because?

S3: Yes, in total, there are twenty portions, because there are four pizzas.

S4: Ok, yes.

S3: And there are six people … ah! They have four … but they are going to be missing pizza [the student performs his calculations mentally].

The students were not able to abstract favorable information when distributing the pizza portions among the six people at table 1 in any of the particular cases. Instead, they chose to relate the situations to another type of representation, which indicates that they performed a folding back and are at the image making level. It is important to highlight that the students in each particular case divided the pizzas on the representation, but this information was eliminated by themselves each time they divided the pizzas (part 16c in **Figure 16**).

S3: Ah! I already know what we are going to do [begins to make another type of pictorial representation]. There are six people, right? This goes to this; this goes to this [makes some lines

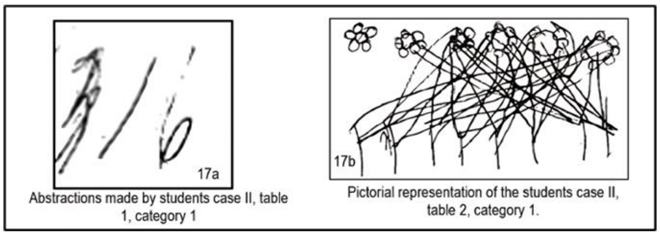

Figure 17. Abstractions and representations made by students case 2, table 1, category 1 (Source: Authors' own elaboration)

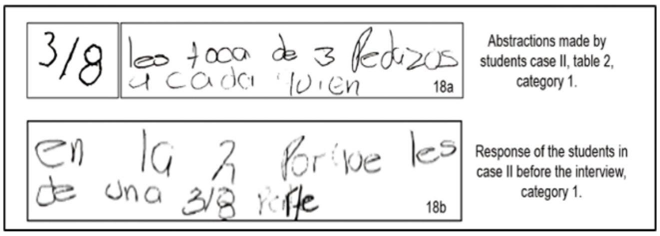

Figure 18. Abstractions and responses made by students in case 2, table 2, category 1 (Source: Authors' own elaboration)

connecting the pieces of pizza with the people in the new representation he makes].

S4: We have one left [they refer to a slice of pizza].

S3: It must be María's, so we have to divide them into six [refers to dividing each pizza into six equal parts].

The students, who had had difficulty abstracting information from their representations, again constructed a pictorial representation of the situation at table 1, including Mary (part 17 a in **Figure 17**). This new representation allowed them to establish the equitable distribution of the pizza portions for the six people (image having level). Following the process of dividing the pizzas by the number of people, they would have a total of twenty-four portions, which implies that each person gets four portions of the pizzas or $\frac{4}{6}$ of each pizza.

S4: We're done [expresses this when finishing distributing the pizzas].

S3: And what is the reflection now?

S4: Three-sixths would go to each person.

S3: Three-sixths?

S4: Yes, because look, we divide each pizza into six parts.

The students were unable to correctly abstract the information. They stated that each person at table 1, including María, was entitled to three portions of pizza if they were divided into six parts, which is the number of people who would be at the table. They expressed their answer with a symbolic representation, the fraction $\frac{3}{6}$, see part 17a in **Figure 17**.

S4: They are three-sixths because the pizzas were divided into six parts and each person would get three portions [following your reasoning, each person would get four portions].

After understanding the situation at table 1, the students tried to apply the same process to table 2. They created a pictorial representation of six pizzas and eight people (image having) but were initially confused as they were not sure if there were seven or eight. people. Then, they began to relate the pizzas to the people using strokes (see part 17b in **Figure 17** and transcript excerpt). However, they continued to have difficulties dividing the pizzas and abstracting information from the process. They failed to reach the level of property noticing, as they did not build specific properties on the image they created, which would have allowed them to generalize.

S3: There are eight people with María.

S4: Mmm no, there are seven.

S3: And Maria?

S4: Oh yes, there are eight people [they add one more person to their representation].

S3: How much does each pizza come with? [refers to the number of servings] Five, right?

S4: Seven portions.

S3: No, seven is the number of people. There are six pizzas, and they are all uncut, so we can divide them as best suits us.

S3: Aren't they divided into seven? [He expresses this when he finishes reading the situation at table 2, but there would be eight including María].

During the interview, the students in case 2 mentioned that the task statement said that the pizzas were uncut, so they could decide how many parts to divide them into. They decided to divide them according to the number of people corresponding to each table. However, despite this decision, they always showed confusion when carrying out the fractionation process. Finally, they managed to abstract information from the image they represented and concluded that each person received three portions of pizza (part 18a in **Figure 18** and transcript extract). Compared to the students in case 1, who used the division process for distribution, the students in case 2 had difficulty abstracting information and generalizing, suggesting that they did not reach the level of property noticing.

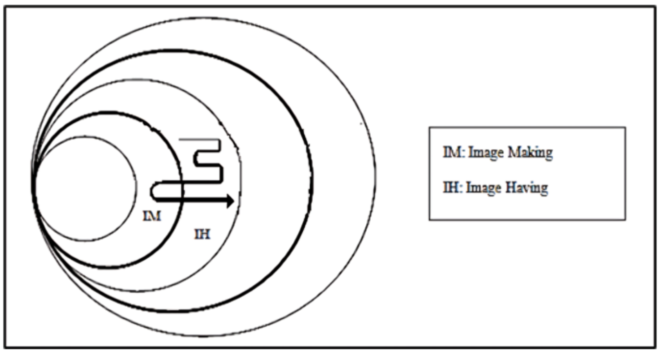

Figure 19. Students' knowledge structure to choose the best ratio, case 2, category 1 (Source: Authors' own elaboration)

S4: How many pizzas were there?

S3: Six pizzas.

S3: Oh, I know, it must be equitable so that each person gets a few portions of each pizza, right?

S4: We cut them into eight [expresses this after carrying out the particular case of dividing each pizza into eight portions].

S4: They are three eighths; they are going to be distributed in three portions and there are eight people.

S3: What helped us is that it said that all the pizzas were uncut, meaning that we could cut them at our convenience.

After distributing the pizzas in relation to the people at both tables, the students had to answer the question of which table they would suggest Maria sit at and what would be the best ratio for it. The students responded by choosing the representation of table 2, which would allow Maria to eat more pizza (part 18b in **Figure 18**), although they did the pizza distributions incorrectly. During the interview, the students discussed the final answer they had given and suggested to Maria that she choose table 1 instead of 2, since at both tables they would eat the same amount of pizza, that is, three portions per person. Since there were more people at table 2, each one would eat less pizza if she chose that option. This indicates that students have difficulties ordering and comparing fractions, since they had selected table 2 only because for them the fraction $\frac{3}{8}$ was greater than $\frac{3}{6}$. This shows that the students have not understood the idea of fraction as a ratio and have not reached the Formalizing level. **Figure 19** schematizes the process followed by the students to determine the choice of the best mathematical ratio (referring to the table where María would eat the most pizzas). The arrows in **Figure 19** indicate that the case studied reached that level in its understanding process, and the thickness of the line represents the development of said process.

Figure 20. Final response from students in case 2, table 1, category 2 (Source: Authors' own elaboration)

Identification of the ratio (category 2)

In relation to question (g), students began to extract information regarding the pictorial representation they had created. According to Pirie and Kieren's (1994) model, students began to develop this question from the image having level, where they work on the image they already possess.

S3: How many pizzas are there?

S3: Four plus two would be six pizzas and you have five people, so it would be one more person that has to be added, so that each person gets a pizza, right? [They do not keep in mind that the statement indicates that Mary chose table 1].

S4: Yes [S4 accepts what was expressed by S3 and only writes what was said by S3 on the worksheet].

During the interview, the students doubted her statement and, maintaining their abstraction process, realized that the task statement told them to assume that Maria had selected table 1. Therefore, six people were initially found and, if two pizzas are added, each should receive one pizza. However, the condition was to maintain the distribution already made and conclude how many people should be added. At this point, the students erased the conclusion of adding only one person (**Figure 20**).

It is evident that the students did not consider the initial condition of maintaining the same distribution that they had already established. This is demonstrated by the lack of identification of the mathematical ratio that is derived from the relationship of pizzas per person. Instead, they generated conclusions based on the idea that only one person should be added (not including Maria), since initially there were five people and four pizzas, and adding two pizzas would leave one person missing to complete six, allowing so each one got a pizza. This indicates that students do not reach the level of Formalizing.

S4: The thing is that if a person is added, they will still have the same thing.

S3: In order for each person to get a pizza while Maria is there, two more people must be added, not one.

S4: Then, two people would be added, they would divide the pizzas equally and they would get the same.

If we continue with the distribution completed by the students of $\frac{3}{6}$ of pizza per person, by adding two pizzas to the table, each divided into six parts, it follows that two people will eat from it. Therefore, to maintain the initial cast, four people would need to be added to the group. This suggests that, in terms of pizza portions, adding two pizzas is actually adding the equivalent amount of pizza to one. The students' conclusion correlates with the idea that, in terms of sharing, the addition of two pizzas is similar to the addition of a single pizza. In relation to question (h), students decide to work with the existing representation and begin to abstract information about the number of pizzas that should be added, following the same approach used for table 1. This shows that they are developing the question from the level of abstraction of image having. It is important to note that, once again, students fail to identify the relationship between the number of people and the amount of pizza.

S4: If there are four friends, they will have to buy one more pizza.

S3: Let's see, there are six pizzas and there is María.

S4: It is a pizza that should be added, because if four friends arrive, each person will get a pizza.

S3: Look, table 2 has seven people and there are six pizzas, so with María there would be eight people and hers would be four of her friends [for a total of twelve people], therefore there would be five pizzas that must be bought.

The students showed confusion when determining how many pizzas they should add to table 2 to maintain the same distribution of pizzas $\frac{3}{8}$ that they had established. Although student S3 presented arguments consistent with the relationship between the pizzas and the people at table 2, student S4 expressed that his reasoning was not adequate. This shows the confusion of the students in each of the processes they carry out, as well as their difficulty in explaining the reasoning used to answer each of the questions in the task.

S4: Wait, that's right, look, each person will have three portions [maintaining the same distribution].

S3: Look, if María chose table 2 and four more friends arrived [read the task statement again] it does not say that she sat at another table.

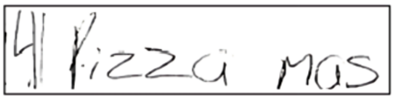

Figure 21. Final student response case 2, table 2, category 2 (Source: Authors' own elaboration)

S4: That's why they buy two more pizzas. There were seven people and with María there would be eight, plus four who arrived.

S4: That's why they distribute the two more pizzas, and they have enough.

S3: Ah, but I said it was so that everyone would get a pizza.

The students are unable to detach themselves from the image they created to make their final conjectures. It is evident that the students did not advance to the Formalizing level. Stating that, if four more people are added to the eight that we initially have (with María), 2 pizzas must be added to table 2 to maintain the same initial distribution $\left(\frac{3}{8}\ of\ pizza\right)$. Furthermore, they present erroneous reasoning by adding two pizzas to table 2 to maintain the same distribution of $\frac{3}{8}$ of pizza. By dividing the eight pizzas into eight parts to distribute them among twelve people, there would be pizzas left over. This shows the difficulty of students in operating and identifying mathematical ratios. Students need to improve their ability to solve problems and ratio mathematically. During the interview, the students show doubt about their statement and distribute the pizzas again, concluding that there would no longer be two more pizzas, but four to add. When carrying out the distribution process following the proportion of $\frac{3}{8}$ of pizza per person, students again have difficulty formalizing their understanding process. This allows us to deduce that the students of case 2 have a weak understanding of the concept of ratio in complex cases (**Figure 21**).

In **Figure 22**, a diagram of the process that the students followed to identify and determine the relationship between the number of pizzas and people on table 1 and table 2 is presented. The arrows indicate that the case under study achieved that level in its process of understanding, and the thickness of the line represents the development of the understanding process.

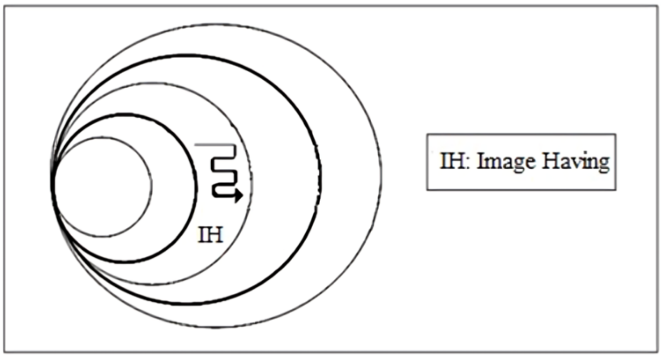

Figure 22. Students' knowledge structure to identify the ratio, case 2, category 2 (Source: Authors' own elaboration)

Distribution strategies (category 3)

The last category of analysis that corresponds to question (i), the students begin by relating the statement with the process of multiplying the seven large tables that the proportion of the statement mentions to them, with the number ten, but they do not clearly explain the ratio for the process or reasoning. This shows that students begin to work on the development of the question from the image making level, where they relate the situation with a mental representation of the situation.

> S4: Ten large tables are needed [the student expresses this after finishing reading the statement without making any type of argument].
>
> S4: We're done, right? [while S3 reread the task statement].
>
> S3: Let's see [checks S4's answer].
>
> S4: Ten tables are needed.

It was observed that students face difficulties in interpreting and addressing the question posed, evidencing a lack of understanding of the concept of ratio and a tendency to get confused in complex situations. In the case of S4, the deduction made does not present solid arguments, and during the interview, the students mentioned that they did not understand the need to multiply seven by ten. This suggests that the students of cases I and II have not developed a good understanding of the concept of ratio in situations where they are presented with complex challenges (**Figure 23**).

> S3: Look we have to place 240 people at large and small tables.
>
> S4: There are eight people so multiply by ten.
>
> S3: But there are two hundred and forty people, you have to find a number that multiplied by eight gives you 240.

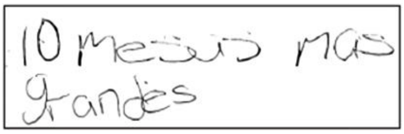

Figure 23. Final student response case 2, category 3 (Source: Authors' own elaboration)

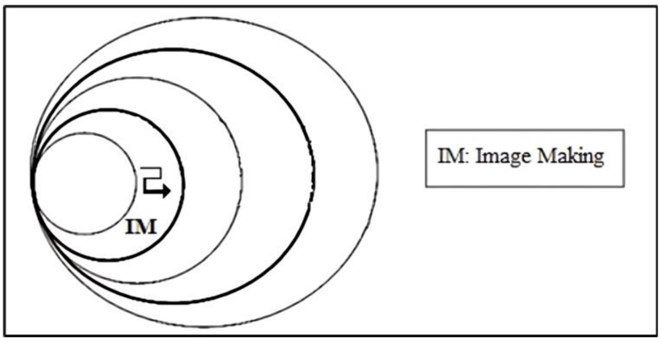

Figure 24. Students' knowledge structure using distribution strategies, case 2, category 3 (Source: Authors' own elaboration)

> S3: Leave it like this [ten large tables are needed].

The students finally abandoned the development of the question and stated that ten large tables are needed to seat 240 people. This indicates that the students have not reached the level of Formalizing. If the students' statement that ten tables are needed is taken as the answer to the question, then only 70 people in total would be accommodated, without maintaining the proportion provided in the statement. **Figure 24** schematizes the process that students followed to determine distribution strategies in situations that involve mathematical ratio. The arrows indicate that the case under study achieved that level in its understanding process, and the thickness of the line represents the development of the understanding process.

Case 1 manages to formalize its understanding process according to Pirie and Kieren (1994) in the identification of the mathematical ratio (category 2). But, in the other two categories of analysis, they will not be able to do so. Case 2 fails to formalize its understanding process in any of the analysis categories, which is due to the difficulties they faced from the beginning in identifying data and arithmetic calculations. Consequently, the students in both cases are still in the process of achieving a conceptual understanding of the concept of ratio and some other mathematical processes.

DISCUSSION

In the present research, the results obtained are of great relevance for the teaching and learning process of mathematics, especially with regard to the understanding of the concept of ratio. Students were able to demonstrate their process of understanding by developing tasks related to this concept of ratio in complex situations, using various mathematical representations and strategies. This will allow teachers to foster a solid understanding of the concept itself.

To understand the concept of ratio in mathematics, it is essential to have prior knowledge. According to Pirie and Kieren's (1994) theory, the understanding of mathematical knowledge is not linear, but rather an iterative construction that involves cycles forward and backward to advance to a higher level. As well as Wahyuningrum et al., (2023), the importance of understanding students' prior knowledge when addressing mathematical concepts is highlighted, since this knowledge influences the assimilation and understanding of new concepts. At the same time as Fauziah and Cahya (2021), it is evident that analyzing students' prior knowledge is crucial for their understanding of the mathematician's language. Therefore, it is essential to consider students' prior knowledge to facilitate the understanding and effective learning of mathematical ratio.

The knowledge structure that students develop when approaching the concept of mathematical ratio is described, as follows: they are made easier to associate the concept with an image (IC) or create it immediately (HI). Although they often have difficulties identifying and relating units or magnitudes, it is highlighted that the most used representations are pictorial and symbolic, with the pictorial representation being the first in which students relate the situations of the task. Students have difficulty working on the image they already have of the concept (PO), which shows the shortcomings they have in using prior knowledge (PK) in relation to the concept of ratio. Finally, it tries to abstract a hypothesis from the results obtained from the implemented strategies, but these are usually erroneous due to the poor processes implemented. Therefore, students are unable to disassociate themselves from the image they created of the mathematical object, which prevents them from formalizing the concept of ratio as a formal object (F).

The study revealed that primary school students have difficulties understanding the concept of ratio, which is reflected in their ability to identify quantities, multiplicative relationships, fraction notation and equivalent fractions. The lack of F of some mathematical concepts and processes becomes an obstacle to understanding new concepts. As Arenas-Peñaloza and Rodríguez-Vásquez (2022) noted, students have difficulties comparing ratios, reading comprehension, data interpretation, and arithmetic processes. It is concluded that students need better understanding. of the concept of ratio and distribution strategies to address complex situations that involve mathematical ratios. It is necessary to develop a higher level of proportional thinking and prior knowledge to improve the mathematics learning process in primary education.

However, various studies on the concept of ratio have been addressed in the literature (Arenas-Peñaloza & Rodríguez-Vásquez, 2022; Fauziah & Cahya, 2021; Wahyuningrum et al., 2023). This research provides the students' written and verbal evidence, as well as the main strategies they used during their comprehension process. In addition, the difficulties they face when trying to formalize the concept of ratio in complex contexts are identified. Other authors (Pouta et al., 2021; Şengül & Kıral Demir, 2024) also underline the importance of continuing to contribute to improving the process of understanding this concept. Therefore, it is essential that teachers get their students to establish connections between their prior knowledge and the mathematical representations and strategies used when developing tasks that promote the understanding of the concept of ratio.

While it is true that there are various investigations that have used different models to analyze the mathematical understanding process of students (Arenas-Peñaloza & Rodríguez-Vásquez, 2022; Cervantes-Barraza et al., 2021; Rodríguez-Nieto et al., 2024), where difficulties, levels of understanding and solution strategies are evident, in the present investigation, in relation to the theoretical model of Pirie and Kieren (1994), it has been determined that students should have reached the level of F throughout the development of the task, but there are still difficulties, which leaves a possibility to continue investigating this mathematical content.

CONCLUSIONS

This research shows that acquiring mathematical understanding of the concept of ratio is not easy, due to the complexity of this mathematical object, therefore, it is essential to continue investigating some problematic cases. For example, in relation to the theoretical model, students should have reached the level of formalizing (F) throughout the development of the task. Therefore, the students in case 1 and case 2 have not adequately understood the concept of ratio in complex situations. To improve the understanding of this concept and address problems, some strategies and approaches can be followed in teaching, such as:

1. **Strengthen students' prior knowledge:** Prior knowledge is the foundation on which learning is built and can be activated and used to facilitate the understanding of new concepts.
2. **Use of mental and visual representations:** These representations can help students understand and solve mathematical problems more effectively.
3. **Promotion of the use of mental models and thinking strategies:** Problem-solving and posing can be used as methodological strategies in mathematics classes to improve understanding of the concept of ratio.

4. **Group discussion and debate:** Encouraging dialogue and group discussion can help students better understand mathematical concepts and identify difficulties in knowing ratio concept.
5. **Incorporating practical activities and exercises into classes:** This may include completing problem-solving exercises, discussing case studies, and solving problems in real situations.

By implementing these strategies, students can improve their understanding of the concept of ratio and address similar challenges more effectively. It is worth noting that one of the benefits of this research is that it shows the written and verbal evidence obtained in the interviews conducted with the students, which was important to account for their understanding together with the Pirie-Kieren theoretical tool. Furthermore, students and teachers could improve their understanding of the concept of ratio when they are aware of the most appropriate procedures to solve a problem step by step or by establishing mathematical connections. That is, future research could emphasize mathematical connections or relationships between concepts, representations, symbols, procedures, etc. (Rodríguez-Nieto et al., 2023, 2024).

Author contributions: JAA-P: conceptualization, methodology, supervision, & writing of the original draft; **AS-S, NN-Y, & AMI-J:** supervision, visualization, & review; **CAR-N:** formal analysis, validation, writing, & review & editing–original draft; **FMR-V:** supervision, visualization, writing–review, editing, & resources. All authors have agreed with the results and conclusions.

Funding: This study was supported by the Universidad de la Costa and the Ministry of Science, Technology and Innovation.

Ethical statement: The authors stated that the participants and the educational institution were informed that this article has educational purposes and not economic or political ones and agreed to participate voluntarily. In addition, one of the researchers is a full-time professor at the institution with research commitments and is authorized to conduct research with the student population with the purpose of improving students' understanding of mathematical concepts.

Declaration of interest: No conflict of interest is declared by the authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

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